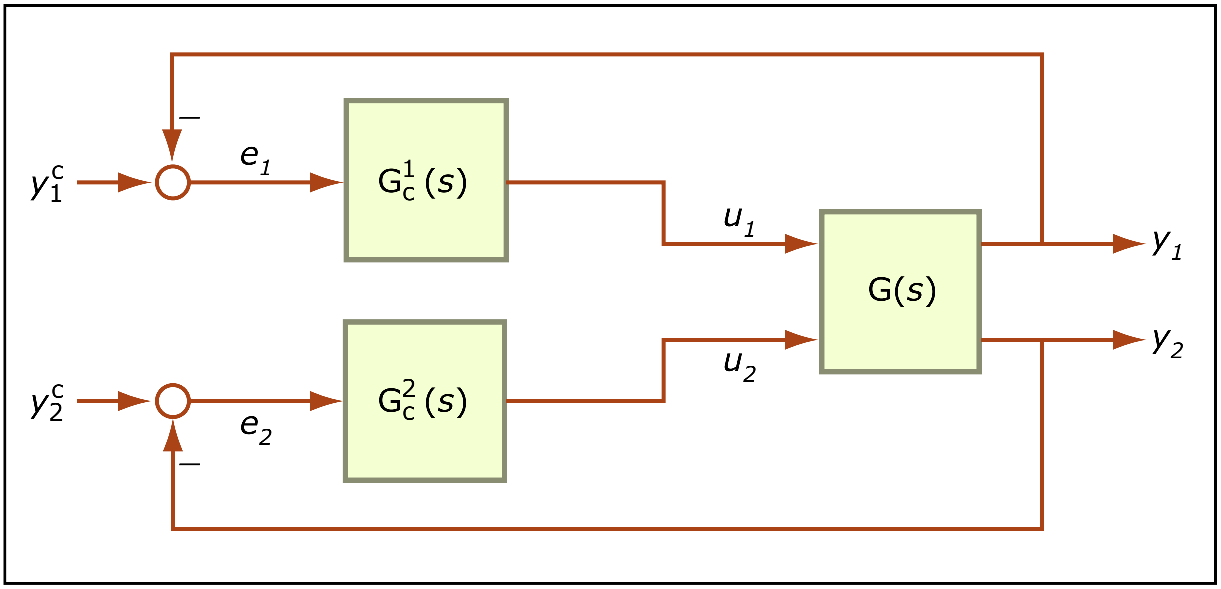
16.30/31, Fall 2010 — Lab #2 Appendix

Consider a system driven by multiple controllers in parallel; a block diagram representing this scenario for two controllers is provided below. The plant and all controllers are specified as state-space models; our objective is to identify the state-space model for the loop dynamics $L(s) = G(s)G_c(s)$, with inputs $(\mathbf{y}_1^c, \mathbf{y}_2^c)$ and outputs $(\mathbf{y}_1, \mathbf{y}_2)$.



The state-space models for the plant and controllers are as follows:

$$G(s): \qquad \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1(t) \\ \mathbf{u}_2(t) \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{y}_1(t) \\ \mathbf{y}_2(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \mathbf{x}(t),$$

$$G_c^1(s): \qquad \dot{\mathbf{x}}_c^1(t) = A_c^1 \mathbf{x}_c^1(t) + B_c^1 \mathbf{e}_1(t),$$

$$\mathbf{u}_1(t) = C_c^1 \mathbf{x}_c^1(t) + D_c^1 \mathbf{e}_1(t),$$

$$G_c^2(s): \qquad \dot{\mathbf{x}}_c^2(t) = A_c^2 \mathbf{x}_c^2(t) + B_c^2 \mathbf{e}_2(t),$$

$$\mathbf{u}_2(t) = C_c^2 \mathbf{x}_c^2(t) + D_c^2 \mathbf{e}_2(t),$$

where $\mathbf{e}_i(t) = \mathbf{y}_i^c(t) - \mathbf{y}_i(t)$. First, form the composite dynamics for the plant $G(s)$ and controller $G_c^1(s)$:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_c^1 \end{bmatrix} = \begin{bmatrix} A & B_1 C_c^1 \\ 0 & A_c^1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c^1 \end{bmatrix} + \begin{bmatrix} B_1 D_c^1 \\ B_c^1 \end{bmatrix} \mathbf{e}_1 + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} \mathbf{u}_2,$$

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c^1 \end{bmatrix}.$$

We can "close the loop" by simply applying the fact that $\mathbf{e}_1(t) = \mathbf{y}_1^c(t) - \mathbf{y}_1(t)$:

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_c^1 \end{bmatrix} &= \begin{bmatrix} A & B_1 C_c^1 \\ 0 & A_c^1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c^1 \end{bmatrix} + \begin{bmatrix} B_1 D_c^1 \\ B_c^1 \end{bmatrix} \left(\mathbf{y}_1^c - \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c^1 \end{bmatrix} \right) + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} \mathbf{u}_2 \\ &= \begin{bmatrix} A - B_1 D_c^1 C_1 & B_1 C_c^1 \\ -B_c^1 C_1 & A_c^1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c^1 \end{bmatrix} + \begin{bmatrix} B_1 D_c^1 \\ B_c^1 \end{bmatrix} \mathbf{y}_1^c + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} \mathbf{u}_2, \\ \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} &= \begin{bmatrix} C_1 & 0 \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c^1 \end{bmatrix}. \end{aligned}$$

By then repeating this process with the second input $\mathbf{u}_2$, only the references $\mathbf{y}_1^c$ and $\mathbf{y}_2^c$ will remain as inputs, yielding the desired state-space model.