2 Weighted-residuals / Galerkin

No need for variational method / principle, i.e. existence of functional $J(u)$ whose stationary points give sought solution (when Vainberg's thm fails).

Start from field equations:

residuals
$$\begin{cases} r_i = \sigma_{ij,j} + f_i = 0 & \text{in } B \\ s_i = t_i - \sigma_{ij} n_j = 0 & \text{on } S_2 \end{cases}$$

Enforce governing equations weakly by weighted averages:

$$\int_B r_i \eta_i \, dV + \int_{S_2} s_i \eta_i \, dS = 0$$

for some suitable collection of weighting functions (admissible variations) η_i.

Weakly $\rightarrow$ weak form (integrate by parts) involves no higher derivatives.

$$\int_B -\sigma_{ij} \eta_{i,j} \, dV + \int_B f_i \eta_i \, dV + \cancel{\overbrace{\int_S \sigma_{ij} \eta_i n_j \, dS}^{A}} + \int_{S_2} t_i \eta_i \, dS - \cancel{A} = 0$$

$S \rightarrow \cancel{S_1} \cup S_2$ $\quad \eta_i$ admissible

$$\boxed{\int_B \sigma_{ij}\,\eta_{i,j}\,dV = \int_{S_2} t_i\,\eta_i\,ds + \int_B f_i\,\eta_i\,dV,\ \forall\,\eta_i \text{ admissi}}$$

Principle of virtual work

- Enforce weak form for $u_i \sim (u_h)_i(x) \in V_h$

$$\eta_i \sim (\eta_h)_i(x) \in W_h$$

i.e. $(u_h)_i(x) = \sum_{a=1}^{N} u_{ia}\,N_a(x)$

$$(\eta_h)_i(x) = \sum_{b=1}^{N} \eta_{ia}\,M_a(x)$$

$M_a = N_a \rightarrow$ Galerkin's method

$$\int_B C_{ijkl}\left(\sum_{b=1}^{N} u_{kb}\,N_{b,l}\right)\left(\sum_{a=1}^{N} \eta_{ia}\,M_{a,j}(x)\right)dV =$$

$$\int_{S_2} t_i\left(\sum_{a=1}^{N} \eta_{ia}\,M_a\right)ds + \int_B f_i\left(\sum_{a=1}^{N} \eta_{ia}\,M_a\right)dV$$

$$\sum_{a=1}^{N} \eta_{ia}\left\{\sum_{b=1}^{N} u_{kb}\left[\int_B C_{ijkl}\,N_{b,l}\,M_{a,j}\,dV\right] - \int_{S_2} t_i\,M_a\,ds\right.$$

$$\left. - \int_B f_i\,M_a\,dV\right\} = 0 \quad \forall\,\eta_{ia} \text{ admiss.}$$

$$\Leftrightarrow \sum_{b=1}^{N} K_{iakb}\, u_{kb} = f_{ia}^{ext}, \quad K u = f$$

$$K_{iakb} = \int_B C_{ijkl}\, N_{b,l}\, M_{a,j}\, dV, \quad K \neq K^T$$

$$f_{ia}^{ext} = \int_{S_2} \bar{t}_i\, M_a\, dS + \int_B f_i\, M_a\, dV$$

Galerkin: $M_a = N_a \longrightarrow K = K^T$

also the stiffness matrix and the external force vector are <u>the same</u> as in Ritz!!

Found the same approximate solution as with constrained minimization of functional (in this case of linear elasticity there is a variational principle)

<u>Geometrical interpretation of Galerkin's Method</u>

<u>Definition</u>: <u>Dirichlet form</u>:

$$a(u, v) = \int_B C_{ijkl}\, u_{k,l}\, v_{i,j}\, dV$$

strain energy $= \frac{1}{2} a(u,u)$

<u>Claim</u>: Dirichlet form defines an inner product over V

<u>Proof</u>: • <u>"a" bilinear</u>: $a(u, \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 a(u,v_1) + \alpha_2 a(u,v_2)$

$a(\alpha_1 u_1 + \alpha_2 u_2, v) = \alpha_1 a(u_1, v) + \alpha_2 a(u_2, v)$

obvious from definition

• <u>Symmetry</u>:

$$a(u,v) = a(v,u)$$

$$\Leftrightarrow C_{ijkl} = C_{klij}$$

• <u>$a(u,u) \geq 0$</u> and $a(u,u) = 0 \Rightarrow u = 0$

requires additional conditions on $C_{ijkl}(x)$

1) $\sup |C_{ijkl}(x)| < C_1 < \infty \quad x \in B$

2) $C_{ijkl}(x) \alpha_{ij} \alpha_{kl} > C_2 |\alpha|^2 \quad \forall x \in B$

In particular: $\alpha_{ij} = \frac{1}{2}(\xi_i \eta_j + \xi_j \eta_i)$

$C_{ijkl} \xi_i \eta_j \xi_k \eta_l > C_2 |\xi|^2 |\eta|^2$

(elasticity operator is uniformly elliptic)

Definition: $u \perp v$ in V if $a(u,v)=0$

Now interpretation:

$$a(u,v) - \langle f, v\rangle = 0 \quad \forall v \in V$$

$$\langle f, v\rangle = \int_{S_2} \bar{t}_i v_i \, ds + \int_B f_i v_i \, dv$$

Galerkin weighting:

① $a(u_h, v_h) - \langle f, v_h\rangle = 0 \quad \forall v_h \in V_h$

In particular: (PVW)

② $a(u, v_h) - \langle f, v_h\rangle = 0 \quad \forall v \in V_h \subset V$

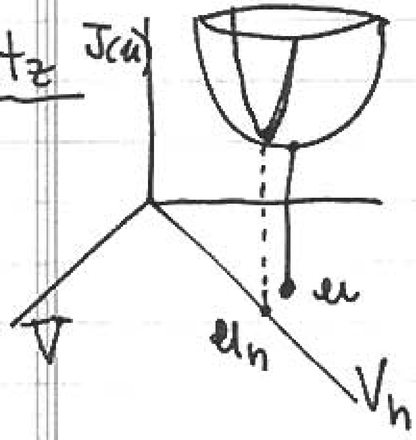
② - ①: $a(u, v_h) - a(u_h, v_h) = 0$

"a" bilinear: $\boxed{a(u-u_h, v_h) = 0}$

$e = u - u_h \equiv$ error $\qquad a(e, v_h) = 0$

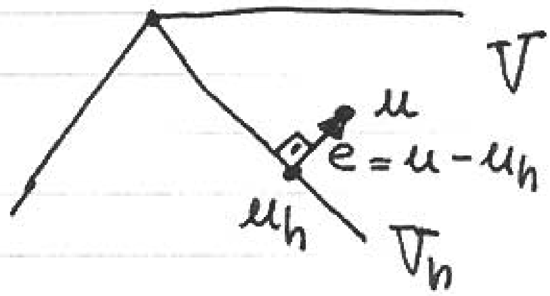
error is orthogonal to all functions in V_h

Ritz



$u_h : J(u_h) = \min_{v \in V_h} J(v)$

Galerkin



- error orthogonal to V_h
- u_h is the projection of u onto V_h

(3) Best approximation method

Definition: Energy norm : $\|u\|_E = \sqrt{a(u,u)}$

$$= \left(\int_B C_{ijkl}\, u_{k,l}\, u_{i,j}\, dV \right)^{1/2}$$

by analogy to vectors $\|v\| = \sqrt{v \cdot v}$

Seek $u_h \in V_h \;/\; \|u - u_h\|_E$ minimum

Minimize $\|u - u_h\|^2 =$

$$= \int_B C_{ijkl} \left(u_{k,l} - \sum_{b=1}^{N} u_{kb} N_{b,l} \right) \left(u_{i,j} - \sum_{a=1}^{N} u_{ia} N_{a,j} \right) dV$$

$$0 = \frac{\partial \| u - u_h \|^2}{\partial u_{ia}} = -2 \int_B C_{ijkl} \left(u_{k,l} - \sum_{b=1}^{N} u_{kb} N_{b,l} \right) N_{a,j} \, dV$$

Still don't know "u". Try to use governing eqns.

$$\sum_{b=1}^{N} \underbrace{\int_B C_{ijkl} N_{b,l} N_{a,j} \, dV}_{K_{iakb}} u_{kb} = \underbrace{\int_B \underbrace{C_{ijkl} u_{k,l}}_{\sigma_{ij}} N_{a,j} \, dV}_{-\int_B \underbrace{\sigma_{ij,j}}_{-f_j} N_a \, dv + \int_{\underbrace{S}_{S_1 \cup S_2}} \sigma_{ij} N_a n_j \, ds}$$

from N_a admissible

$$\Rightarrow \quad K \, U = f^{ext}$$

same "K" !! again !!.

- $e \perp V_h$
- u_h closest element in V_h to u in V_h.

→ EQUIVALENT

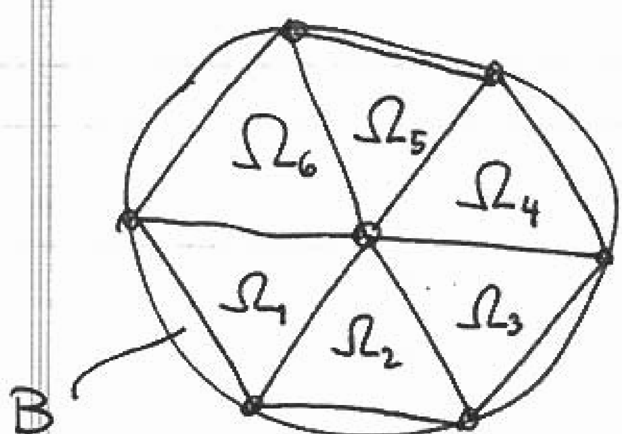
Summary: Approximate solution $u_h = \sum u_a N_a$ follows from:

1. Constrained minimization (Rayleigh-Ritz)
2. Weak form + weight functions $\in V_h$ (Galerkin)
3. Minimizing energy distance to exact solution (best approximation)

The finite element method

Want to formulate convenient shape functions N_a.

1. Partition $B = \bigcup_{e=1}^{E} \Omega_h^e$



$\{\Omega_h^e\}$ pairwise disjoint

(2) Use local polynomial interpolation

u_h: approximate solution

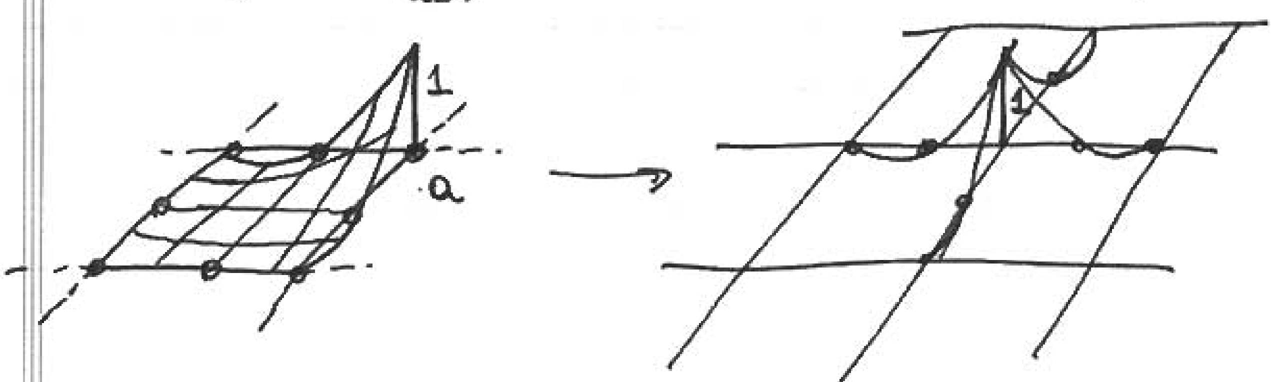
u_h^e: restriction of u_h to Ω_h^e

$$u_h^e(x) = \begin{cases} u_h(x) & \text{if } x \in \Omega_h^e \\ 0 & \text{otherwise} \end{cases}$$

$$u_h(x) = \underbrace{\sum_{e=1}^{E}}_{\text{Assembly operator}} u_h^e(x)$$

Introduce a set of local interpolation functions $N_a^e(x)$ defined over Ω_h^e and set:

$$u_h^e(x) = \sum_{a=1}^{n} u_a^e N_a^e(x) \quad , \; n: \text{\# of nodes/element}$$



If x_b^e are the coordinates of the nodes of element Ω_h^e, require:

- $N_a^e(x_b^e) = \delta_{ab}$ /

$$(u_h^e)_i(x_b^e) = \sum_{a=1}^{n} N_a^e(x_b^e)\, u_{ia}^e = u_{ib}$$

→ "u_{ib}" are the nodal displacements

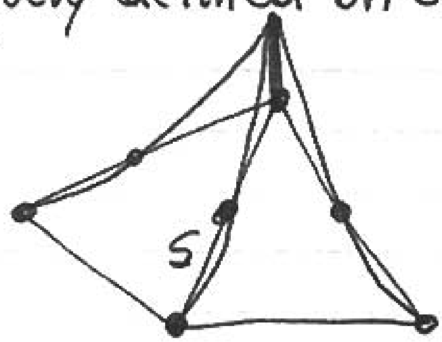
Example: 1D Lagrange polynomials

Continuity requirements: Global FE displacements u_h must be in $H_0^1(B)$

Sufficient conditions:

- N_a^e must be $C^1(\Omega_h^e)$
- N_a, global shape functions obtained by piecing together local shapefunctions N_a^e must be $C^0 \Longrightarrow$ Shape function N_a^e must be

uniquely defined on sides.



restriction of u_h^e to S is determined uniquely by nodal values on S.

$$u_h(x) = \sum_{e=1}^{E} u_h^e(x)$$

Introduce a global numbering of the mesh nodes

$a = 1, ---, N$

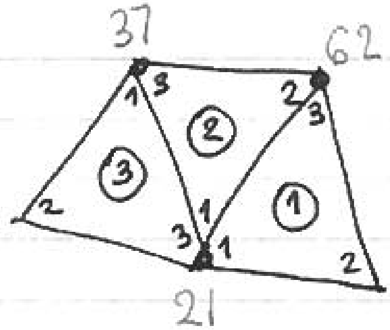
<u>Connectivity table</u>: (summit::connectivity)

$a = g(b, e)$

a: global # of the node $\quad 1, ---, N$

b: local # of the node $\quad 1, ---, n$

e: element number $\quad 1, ---, E$



$g(2,3) = 37$