16.225 - Computational Mechanics of Materials
Homework assignment # 4
Handed out: 11/3/03
Due: 11/17/03

Consider a nonlinear elastic solid described by the strain energy density:

$$W = k[J \log(J) - J] + \frac{\mu}{2} \text{tr}\mathbf{C}^{\text{dev}}$$

where

$$\mathbf{C}^{\text{dev}} = J^{-2/3}\mathbf{C}$$

and k and μ are material constants. For this model material,

1. Derive the expressions for the components of
   (a) the first and second PK stress tensors, P_{iJ} and S_{IJ}; the Cauchy stress tensor σ_{ij}; and the Kirchoff stress tensor τ_{ij}.
   (b) The material moduli C_{IJKL}; the Lagrangian moduli C_{iJkL}; and the spatial moduli C_{ijkl}.

2. Implement this constitutive model in sumMIT within the framework proposed in HA01. The input to the subroutine should be the deformation gradients and the material properties; the output the strain energy density, the first PK stresses and the Lagrangian tangent moduli. Conduct the consistency test of your implementation of this material model. Carry out the test both for trivial ($\mathbf{F} = \mathbf{I}$, where $\mathbf{I}$ is the identity tensor), and for nontrivial—but reasonable—deformation gradients $\mathbf{F}$.

3. Extend your 6-noded isoparametric triangular element to nonlinear elasticity. The element should evaluate:

(a) the (incomplete) out-of-balance force array (isw=6),

$$r_{ia} = \int_{\Omega_0^e} \rho_0 B_i N_a dV_0 - \int_{\Omega_0^e} P_{iI} N_{a,I} dV_0$$

which includes gravity loads, $B_i = g_i = constant$, but excludes external surface tractions since these are imposed globallyf

(b) the tangent stiffness matrix (isw=3),

$$K_{iakb} = \int_{\Omega_0^e} C_{iIjJ} N_{b,J} N_{a,I} dV_0$$

The correctness of the tangent stiffness matrix should be tested by numerical differentiation, i.e. a consistency test of the implementation of the tangent stiffness matrix in terms of the out-of-balance force array:

$$K_{iakb} = -\frac{\partial r_{ia}^{int}}{\partial x_{kb}}$$

4. As an example of application, consider a square domain $[0, L]^2$ subjected to loads $\bar{T}_1 = T$, $\bar{T}_2 = 0$ on edge $X_1 = L$, fully supported $\phi_1 = 0$, $\phi_2 = X_2$, on edge $X_1 = 0$. Increase the load parameter T monotonically from 0 to $5\mu L$ in 5 equal load increments. Solve the equilibrium equations by a Newton-Raphson iteration with $TOL = 10^{-9}$. Plot the deformed meshes for each of the five load increments, and the load-displacement curve.