16.21 Techniques of Structural Analysis and Design
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Unit #9 - Calculus of Variations

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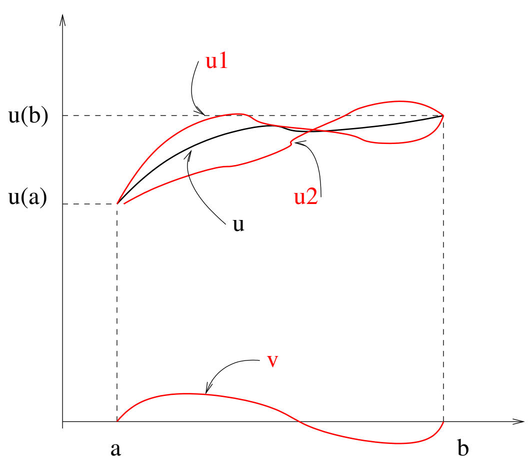
Let u be the actual configuration of a structure or mechanical system. u satisfies the displacement boundary conditions: $u = u^*$ on S_u. Define:

$$\bar{u} = u + \alpha v, \quad \text{where:}$$
$$\alpha : \text{ scalar}$$
$$v : \text{ arbitrary function such that } v = 0 \text{ on } S_u$$

We are going to define αv as δu, the *first variation* of u:

$$\boxed{\delta u = \alpha v} \tag{1}$$

Schematically:



As a first property of the *first variation*:

$$\frac{d\bar{u}}{dx} = \frac{du}{dx} + \alpha\underbrace{\frac{dv}{dx}}$$

so we can identify $\alpha\frac{dv}{dx}$ with the first variation of the derivative of u:

$$\delta\Big(\frac{du}{dx}\Big) = \alpha\frac{dv}{dx}$$

But:

$$\alpha\frac{dv}{dx} = \frac{\alpha dv}{dx} = \frac{d}{dx}(\delta u)$$

We conclude that:

$$\boxed{\delta\Big(\frac{du}{dx}\Big) = \frac{d}{dx}(\delta u)}$$

Consider a function of the following form:

$$F = F(x, u(x), u'(x))$$

It depends on an independent variable x, another function of x ($u(x)$) and its derivative ($u'(x)$). Consider the change in F, when u (therefore u') changes:

$$\begin{aligned}\Delta F &= F(x, u + \delta u, u' + \delta u') - F(x, u, u') \\ &= F(x, u + \alpha v, u' + \alpha v') - F(x, u, u')\end{aligned}$$

expanding in Taylor series:

$$\begin{aligned}\Delta F = F + \frac{\partial F}{\partial u}\alpha v + \frac{\partial F}{\partial u'}\alpha v' + \frac{1}{2!}\frac{\partial^2 F}{\partial u^2}(\alpha v)^2 + \frac{1}{2!}\frac{\partial^2 F}{\partial u \partial u'}(\alpha v)(\alpha v') + \cdots - F \\ = \frac{\partial F}{\partial u}\alpha v + \frac{\partial F}{\partial u'}\alpha v' + h.o.t.\end{aligned}$$

First total variation of F:

$$\begin{aligned}\delta F = \alpha\Big[\lim_{\alpha\to 0}\frac{\Delta F}{\alpha}\Big] \\ = \alpha\lim_{\alpha\to 0}\Big[\frac{F(x, u + \alpha v, u' + \alpha v') - F(x, u, u')}{\alpha}\Big] \\ = \alpha\lim_{\alpha\to 0}\Big[\frac{\frac{\partial F}{\partial u}\alpha v + \frac{\partial F}{\partial u'}\alpha v'}{\alpha}\Big] = \frac{\partial F}{\partial u}\alpha v + \frac{\partial F}{\partial u'}\alpha v'\end{aligned}$$

$$\boxed{\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'}$$

Note that:

$$\delta F = \alpha \frac{dF(x, u + \alpha v, u' + \alpha v')}{d\alpha}\Big|_{\alpha=0}$$

since:

$$\frac{dF(x, u + \alpha v, u' + \alpha v')}{d\alpha} = \frac{\partial F(x, u + \alpha v, u' + \alpha v')}{\partial u} v + \frac{\partial F(x, u + \alpha v, u' + \alpha v')}{\partial u'} v'$$

evaluated at $\alpha = 0$

$$\frac{dF(x, u + \alpha v, u' + \alpha v')}{d\alpha}\Big|_{\alpha=0} = \frac{\partial F(x, u, u')}{\partial u} v + \frac{\partial F(x, u, u')}{\partial u'} v'$$

Note analogy with differential calculus.

$$\delta(aF_1 + bF_2) = a\delta F_1 + b\delta F_2 \quad \text{linearity}$$
$$\delta(F_1 F_2) = \delta F_1 F_2 + F_1 \delta F_2$$
$$\text{etc}$$

The conclusions for $F(x, u, u')$ can be generalized to functions of several independent variables x_i and functions u_i, $\frac{\partial u_i}{\partial x_j}$:

$$F\Big(x_i, u_i, \frac{\partial u_i}{\partial x_j}\Big)$$

We will be making intensive use of these properties of the *variational operator* δ:

$$\frac{d}{dx}(\delta u) = \frac{d}{dx}(\alpha v) = \alpha \frac{dv}{dx} = \delta\Big(\frac{du}{dx}\Big)$$
$$\int \delta u dx = \int \alpha v dx = \alpha \int v dx = \delta\Big(\int u dx\Big)$$

Concept of a functional

$$I(u) = \int_a^b F(x, u(x), u'(x)) dx$$

First variation of a functional:

$$\delta I = \delta\Big(\int F(x, u(x), u'(x))dx\Big)$$
$$= \int \delta\Big(F(x, u(x), u'(x))\Big)dx$$
$$\boxed{\delta I = \int \Big(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\Big)dx}$$

Extremum of a functional

"u_0" is the *minumum* of a functional if:

$$\boxed{I(u) \geq I(u_0)\forall u}$$

A necessary condition for a functional to attain an extremum at "u_0" is:

$$\boxed{\delta I(u_0) = 0,\ \text{or}\ \ \frac{dI}{d\alpha}(u_0 + \alpha v, u_0' + \alpha v')\Big|_{\alpha=0} = 0}$$

Note analogy with differential calculus. Also difference since here we require $\frac{dF}{d\alpha} = 0$ at $\alpha = 0$.

$$\delta I = \int_a^b \Big(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\Big)dx$$

Integrate by parts the second term to get rid of $\delta u'$.

$$\delta I = \int_a^b \Big[\frac{\partial F}{\partial u}\delta u + \frac{d}{dx}\Big(\frac{\partial F}{\partial u'}\delta u\Big) - \delta u\frac{d}{dx}\Big(\frac{\partial F}{\partial u'}\Big)\Big]dx$$
$$= \int_a^b \Big[\frac{\partial F}{\partial u} - \frac{d}{dx}\Big(\frac{\partial F}{\partial u'}\Big)\Big]\delta u dx + \frac{\partial F}{\partial u'}\delta u\Big|_a^b$$

Require δu to satisfy homogeneous displacement boundary conditions:

$$\delta u(b) = \delta u(a) = 0$$

Then:

$$\delta I = \int_a^b \Big[\frac{\partial F}{\partial u} - \frac{d}{dx}\Big(\frac{\partial F}{\partial u'}\Big)\Big]\delta u dx = 0,$$

$\forall \delta u$ that satisfy the appropriate differentiability conditions and the homogeneous essential boundary conditions. Then:

$$\boxed{\frac{\partial F}{\partial u} - \frac{d}{dx}\Big(\frac{\partial F}{\partial u'}\Big) = 0}$$

These are the *Euler-Lagrange equations* corresponding to the variational problem of finding an extremum of the functional I.

Natural and essential boundary conditions A weaker condition on δu also allows to obtain the Euler equations, we just need:

$$\frac{\partial F}{\partial u'}\delta u\Big|_a^b = 0$$

which is satisfied if:

- $\delta u(a) = 0$ and $\delta u(b) = 0$ as before
- $\delta u(b) = 0$ and $\frac{\partial F}{\partial u'}(b) = 0$
- $\frac{\partial F}{\partial u'}(a) = 0$ and $\delta u(b) = 0$
- $\frac{\partial F}{\partial u'}(a) = 0$ and $\frac{\partial F}{\partial u'}(b) = 0$

Essential boundary conditions: $\delta u\big|_{S_u} = 0$, or $u = u_0$ on S_u
Natural boundary conditions: $\frac{\partial F}{\partial u'} = 0$ on S.

Example: Derive Euler's equation corresponding to the total potential energy functional $\Pi = U + V$ of an elastic bar of length L, Young's modulus E, area of cross section A fixed at one end and subject to a load P at the other end.

$$\Pi(u) = \int_0^L \frac{EA}{2}\Big(\frac{du}{dx}\Big)^2 dx - Pu(L)$$

Compute the first variation:

$$\delta\Pi = \int \frac{EA}{\cancel{2}}\, \cancel{2}\frac{du}{dx}\delta\Big(\frac{du}{dx}\Big)dx - P\delta u(L)$$

Integrate by parts

$$\delta\Pi = \Big[\frac{d}{dx}\Big(EA\frac{du}{dx}\delta u\Big) - \frac{d}{dx}\Big(EA\frac{du}{dx}\Big)\delta u\Big]dx - P\delta u(L)$$

$$= -\int_0^L \delta u\frac{d}{dx}\Big(EA\frac{du}{dx}\Big)dx + EA\frac{du}{dx}\delta u\Big|_0^L - P\delta u(L)$$

Setting $\delta\Pi = 0, \forall\ \delta u\ /\ \delta u(0) = 0$:

$$\boxed{\frac{d}{dx}\Big(EA\frac{du}{dx}\Big) = 0}$$

$$P = EA\frac{du}{dx}\Big|_L$$

Extension to more dimensions

$$I = \int_V F(x_i, u_i, u_{i,j})dV$$

$$\delta I = \int_V \Big(\frac{\partial F}{\partial u_i}\delta u_i + \frac{\partial F}{\partial u_{i,j}}\delta u_{i,j}\Big)dV$$

$$= \int_V \Big[\frac{\partial F}{\partial u_i}\delta u_i + \frac{\partial}{\partial x_j}\Big(\frac{\partial F}{\partial u_{i,j}}\delta u_i\Big) - \frac{\partial}{\partial x_j}\Big(\frac{\partial F}{\partial u_{i,j}}\Big)\delta u_i\Big]dV$$

Using divergence theorem:

$$\delta I = \int_V \Big[\frac{\partial F}{\partial u_i} - \frac{\partial}{\partial x_j}\Big(\frac{\partial F}{\partial u_{i,j}}\Big)\Big]\delta u_i dV + \int_S \frac{\partial F}{\partial u_{i,j}}\delta u_i n_j dS$$

Extremum of functional I is obtained when $\delta I = 0$, or when:

$$\boxed{\frac{\partial F}{\partial u_i} - \frac{\partial}{\partial x_j}\Big(\frac{\partial F}{\partial u_{i,j}}\Big) = 0}, \text{ and}$$

$$\delta u_i = 0 \ \text{ on } S_u$$

$$\frac{\partial F}{\partial u_{i,j}}n_j \ \ \text{on} S - S_u = S_t$$

The boxed expressions constitute the Euler-Lagrange equations corresponding to the variational problem of finding an extremum of the functional I.