16.21 Techniques of Structural Analysis and Design
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Unit #8 - Principle of Virtual Forces

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Principle of Virtual Forces (PVF)

Consider a compatible displacement field u_i in a deforming structure B of volume V and its associated strain field:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

The displacement field necessarily satisfies the displacement (essential) boundary conditions:

$$u_i = u_i^* \text{ on } S_u \tag{2}$$

where u_i^* are the imposed displacements on the displacement part of the boundary S_u. We multiply (1) by an somewhat arbitrary *virtual stess field* $\bar{\sigma}_{ij}$ to be further defined below and integrate over the volume of the material.

$$\int_V \left[\epsilon_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})\right]\bar{\sigma}_{ij} dV = 0 \tag{3}$$

In addition, we multiply (2) by a set of somewhat arbitrary boundary *virtual tractions* $\bar{t}_i$:

$$\int_{S_u} (u_i - u_i^*)\bar{t}_i dS = 0 \tag{4}$$

Adding up (3) and (4) we obtain:

$$\int_V \Big[\epsilon_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})\Big]\bar{\sigma}_{ij} dV + \int_{S_u} (u_i - u_i^*)\bar{t}_i dS = 0 \tag{5}$$

We will require the virtual stress field and the external virtual tractions to be self-equilibrated, i.e.,

$$\bar{\sigma}_{ji,j} + \bar{f}_i = 0 \text{ in } B \tag{6}$$

$$\bar{t}_i = \bar{\sigma}_{ij} n_j \text{ on } S_t \tag{7}$$

where $\bar{f}_i$ is the set of virtual body forces neccesary to internally equilibrate the virtual stresses. From (5):

$$\int_V \epsilon_{ij}\bar{\sigma}_{ij} dV - \int_V u_{i,j}\bar{\sigma}_{ij} dV + \int_{S_u} (u_i - u_i^*)\bar{\sigma}_{ij} n_j dS = 0 \tag{8}$$

Making use of the now conventional tools of integration by parts and divergence theorem:

$$\int_V \epsilon_{ij}\bar{\sigma}_{ij} dV - \int_V \Big[\big(u_i\bar{\sigma}_{ij}\big)_{,j} - u_i\bar{\sigma}_{ij,j}\Big] dV + \int_{S_u} (u_i - u_i^*)\bar{\sigma}_{ij} n_j dS = 0 \tag{9}$$

$$\int_V \epsilon_{ij}\bar{\sigma}_{ij} dV - \int_S u_i\bar{\sigma}_{ij} n_j dS + \int_V u_i\bar{\sigma}_{ij,j} dV + \int_{S_u} (u_i - u_i^*)\bar{\sigma}_{ij} n_j dS = 0 \tag{10}$$

The second integral may be reduced to the integral over S_u as the virtual stresses are admissible ($\bar{t}_i = \bar{\sigma}_{ij} n_j = 0$ on S_t). The resulting integral over S_u cancels with the fourth term. Using (6) and (7) results in:

$$\int_V \epsilon_{ij}\bar{\sigma}_{ij} dV + \int_V u_i(-\bar{f}_i) dV - \int_{S_u} u_i^*\bar{t}_i dS = 0 \tag{11}$$

or:

$$\boxed{\int_V \epsilon_{ij}\bar{\sigma}_{ij} dV = \int_V u_i\bar{f}_i dV + \int_{S_u} u_i^*\bar{t}_i dS, \ \forall \text{ admissible virtual stress fields}} \tag{12}$$

which is the expression of the *Principle of virtual forces.*

Unit dummy load method

Provides a powerful tool to determine relations between forces and displacements in structures. If $\mathbf{u}_0$ is the actual displacement at the point of interest, we prescribe a virtual concentrated force $\bar{\mathbf{R}}_0$ at that point in the direction of the displacement. This virtual force induces a virtual stress field in the structure which is assumed in self equilibrium. The PVF then reads:

$$u_0 R_0 = \int_V \epsilon_{ij} \bar{\sigma}_{ij} dV \tag{13}$$

In particular R_0 may adopt the value 1 as there is no restriction of this kind on the virtual force field.

Example:

- Specialize the PVF for simple beam theory.

$$u_3(x_1^*)\bar{P} = \int_V (-x_3 u_3'') \frac{-\bar{M} x_3}{I} dx_1 = \int_0^L \frac{M}{EI} \bar{M} dx_1 \tag{14}$$

- Use a virtual force $\bar{P}$ at point x_1^* to determine the value of the displacement at the tip of a cantilever beam loaded with a tip load P. using the PVF expression obtained.

  The actual moment distribution corresponding to the load P in a cantilever beam is $M = P(L - x_1)$, and the virtual moment distribution corresponding to the virtual load $\bar{P}$ has the same form in this case $\bar{M} = \bar{P}(L - x_1)$. The PVF specializes to:

$$u_3(L)\bar{P} = \int_0^L \frac{P(L - x_1)}{EI} \bar{P}(L - x_1) dx_1 \tag{15}$$

  Eliminating $\bar{P}$ (PVF holds for arbitrary values of it) and computing the integral:

$$u_3(L) = \int_0^L \frac{P(L - x_1)}{EI} \bar{P}(L - x_1) dx_1 \tag{16}$$

$$u_3(L) = \frac{PL^3}{3EI} \tag{17}$$
