

16.21 Techniques of Structural Analysis and  
Design  
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Unit #8 - Principle of Virtual Displacements

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**Principle of Virtual Displacements**

Consider a body in equilibrium. We know that the stress field must satisfy the differential equations of equilibrium. Multiply the differential equations of equilibrium by an “arbitrary” displacement field  $\bar{u}_i$ :

$$(\sigma_{ji,j} + f_i)\bar{u}_i = 0 \quad (1)$$

Note that the field  $\bar{u}_i$  is NOT the actual displacement field  $u_i$  corresponding to the solution of the problem but a *virtual* displacement field. Therefore, equation (1) can be interpreted as the local expression of *virtual work* done by the *actual stresses* and the body forces on the *virtual displacement*  $\bar{u}_i$  and that it must be zero. The total virtual work done on the body is obtained by integration over the volume:

$$\int_V (\sigma_{ji,j} + f_i)\bar{u}_i dV = 0 \quad (2)$$

and it must also be zero since the integrand is zero everywhere in the domain.

$$\int_V \sigma_{ji,j} \bar{u}_i dV + \int_V f_i \bar{u}_i dV = 0 \quad (3)$$

$$\int_V [(\sigma_{ji} \bar{u}_i)_{,j} - \sigma_{ji} \bar{u}_{i,j}] dV + \int_V f_i \bar{u}_i dV = 0 \quad (4)$$

$$\int_S \sigma_{ji} \bar{u}_i n_j dS - \int_V \sigma_{ij} \bar{\epsilon}_{ij} dV + \int_V f_i \bar{u}_i dV = 0 \quad (5)$$

The integral over the surface can be decomposed into two: an integral over the portion of the boundary where the actual external surface loads (tractions) are specified  $S_t$  and an integral over the portion of the boundary where the displacements are specified (supports)  $S_u$ . This assumes that these sets are disjoint and complementary, i.e.,

$$S = S_u \cup S_t, \quad S_u \cap S_t = \emptyset \quad (6)$$

$$\int_{S_t} t_i \bar{u}_i dS + \int_{S_u} \sigma_{ji} \bar{u}_i n_j dS - \int_V \sigma_{ij} \bar{\epsilon}_{ij} dV + \int_V f_i \bar{u}_i dV = 0 \quad (7)$$

We will require that the virtual displacements  $\bar{u}_i$  vanish on  $S_u$ , i.e., that the *virtual displacement field* satisfy the *homogeneous essential boundary conditions*:

$$\bar{u}_i(x_j) = 0, \quad \text{if } x_j \in S_u \quad (8)$$

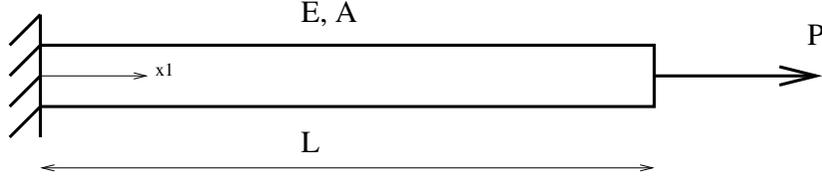
Then, the second integral vanishes. The resulting expression is a statement of the *Principle of Virtual Displacements* (PVD):

$$\boxed{\int_V \sigma_{ij} \bar{\epsilon}_{ij} dV = \int_{S_t} t_i \bar{u}_i dS + \int_V f_i \bar{u}_i dV} \quad (9)$$

It reads: *The work done by the external tractions and body forces on an **admissible** (differentiable and satisfying the homogeneous boundary conditions but otherwise arbitrary) displacement field is equal to the work done by the equilibrated stresses (the actual solution of the problem) on the virtual strains (the strains produced by the virtual field).*

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**Example:** Consider the bar under a tensile load shown in the figure:



The PVD applied to this case is:

$$\int_V \sigma_{11} \frac{d\bar{u}_1}{dx_1} dV = P\bar{u}_1 \Big|_{x_1=L}$$

$$A \int_0^L E \frac{du_1}{dx_1} \frac{d\bar{u}_1}{dx_1} dx_1 = P\bar{u}_1 \Big|_{x_1=L}$$

$$EA \int_0^L \left[ \frac{d}{dx_1} \left( \frac{du_1}{dx_1} \bar{u}_1 \right) - \frac{d^2 u_1}{dx_1^2} \bar{u}_1 \right] dx_1 = P\bar{u}_1 \Big|_{x_1=L}$$

$$\left[ EA \frac{du_1}{dx_1} \bar{u}_1 \right]_{x_1=L} - \left[ EA \frac{du_1}{dx_1} \bar{u}_1 \right]_{x_1=0} - EA \int_0^L \frac{d^2 u_1}{dx_1^2} \bar{u}_1 dx_1 = P\bar{u}_1 \Big|_{x_1=L}$$

The second term on the left hand side is zero because we have asked that  $\bar{u}_1 = 0$  at the support. Note we have not asked for any condition on  $\bar{u}_1$  at  $x_1 = L$  where the load is applied.

$$\left[ EA \frac{du_1}{dx_1} \Big|_{x_1=L} - P \right] \bar{u}_1 \Big|_{x_1=L} = EA \int_0^L \frac{d^2 u_1}{dx_1^2} \bar{u}_1 dx_1$$

The only way this expression can be satisfied for any *admissible* virtual displacement field  $\bar{u}_1$  is if:

$$P = EA \frac{du_1}{dx_1} \Big|_{x_1=L}$$

and

$$EA \frac{d^2 u_1}{dx_1^2} = 0$$

which represent the equilibrium conditions at the boundary and inside the bar, respectively:

$$P = A \left( E \frac{du_1}{dx_1} \right) \Big|_{x_1=L} = A \sigma_{11} \Big|_{x_1=L}$$

and

$$\frac{d}{dx_1} \left( EA \frac{du_1}{dx_1} \right) = \frac{d}{dx_1} \sigma_{11} = 0$$

The solution of this problem is:

$$u_1(x_1) = ax_1 + b$$

the boundary conditions are:

$$u_1(0) = 0 \Rightarrow b = 0$$

$$\frac{P}{A} = Ea$$

$$u_1 = \frac{P}{EA}x_1$$

$$\epsilon_{11} = \frac{du_1}{dx_1} = \frac{P}{EA}$$

$$\sigma_{11} = E\epsilon_{11} = \frac{P}{A}$$

**Example:** With the exact solution of the problem of the bar under a tensile load, verify the satisfaction of the PVD for the following virtual displacement fields:

- $\bar{u}_1 = ax_1$ :

$$AE \int_0^L \frac{P}{EA} a dx_1 = PaL(?)$$

$$PaL = PaL \text{ q.e.d.}$$

- $\bar{u}_1 = ax_1^2$ :

$$AE \int_0^L \frac{P}{EA} 2ax_1 dx_1 = PaL^2(?)$$

$$\frac{A}{E} \frac{EP}{A} 2a \frac{L^2}{2} = PaL \text{ q.e.d.}$$

**Remarks:**

- Principle of Virtual Displacements:
  - enforces equilibrium (in weak form)
  - enforces traction (natural) boundary conditions
  - does NOT enforce displacement (essential) boundary conditions
  - will be satisfied for all equilibrated solutions, compatible or incompatible

## Unit dummy displacement method

Another application of the PVD: provides a way to compute reactions (or displacements) in structures **directly from PVD**. Consider the concentrated reaction force at point "0" of a structure in equilibrium under a set of loads and supports. We can prescribe an arbitrary admissible displacement field  $\bar{u}_i$  and the PVD will hold. The *unit dummy displacement method* consists of choosing the virtual displacement field such that  $\bar{u}_i(\mathbf{x}_0) = 1$  in the direction of the reaction  $R_0$  we are interested in. Then the virtual work of the reaction is  $\bar{\mathbf{u}}_0 \cdot \mathbf{R}_0 = R_0$ . The PVD then reads (in the absence of body forces):

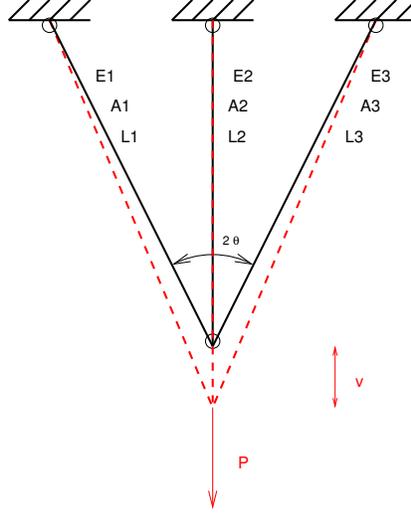
$$\mathbf{R}_0 \cdot \bar{\mathbf{u}}_0 = \int_V \sigma_{ij} \bar{\epsilon}_{ij} dV \quad (10)$$

$$\boxed{R_0 = \int_V \sigma_{ij} \bar{\epsilon}_{ij} dV} \quad (11)$$

where  $\bar{\epsilon}_{ij}$  are the *virtual strains* produced by the *virtual displacement field*  $\bar{\mathbf{u}}_0$ .

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**Example:**



Different materials and areas of cross section:  $E_1, E_2, E_3, A_1, A_2, A_3$ , but require symmetry to simplify the problem:  $E_3 = E_1, A_3 = A_1$ . For a truss element:  $\sigma = E\epsilon$  (uniaxial state).

$$P\bar{v} = A_1L_1\sigma_1\bar{\epsilon}_1 + A_2L_2\sigma_2\bar{\epsilon}_2 + A_3L_3\sigma_3\bar{\epsilon}_3$$

Note: the indices in these expressions just identify the truss element number. The goal is to provide expressions of the virtual strains  $\bar{\epsilon}_I$  in terms of the virtual displacement  $\bar{v}$  so that they cancel out. From the figure, the strains ensued by the truss elements as a result of a tip displacement  $v$  are:

$$\begin{aligned} \epsilon_1 = \epsilon_3 &= \frac{\sqrt{(L_2 + v)^2 + (L_2 \tan \theta)^2} - L_1}{L_1} \\ &= \frac{\sqrt{L_2^2(1 + \tan^2 \theta) + 2L_2v + v^2} - L_1}{L_1} \end{aligned}$$

neglecting the higher order term  $v^2$  and using  $1 + \tan^2 \theta = 1 + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$  we obtain:

$$\epsilon_1 = \epsilon_3 = \frac{\sqrt{\frac{L_2^2}{\cos^2 \theta} + 2L_2v} - L_1}{L_1} = \frac{L_1 \sqrt{1 + \frac{2L_2v}{L_1^2}} - L_1}{L_1} = \sqrt{1 + \frac{2L_2v}{L_1^2}} - 1$$

where we have made use of the fact that:  $\frac{L_2}{\cos \theta} = L_1$ . We seek to extract the linear part of this strain, which should have a linear dependence on the

displacement  $v$ . This can be done by doing a Taylor series expansion of the square root term  $\sqrt{1+2x} = 1 + x + O[x]^2$  (Mathematica tip: Taylor series expansions can be obtained by using the `Series` function. In this case: `Series[Sqrt[1 + 2x], x, 0, 3]`).

$$\epsilon_1 = \epsilon_3 = 1 + \frac{L_2}{L_1^2}v - 1 = \frac{L_2}{L_1^2}v$$

which is the sought expression. The expression for  $\epsilon_2$  can be obtained in a much more straightforward manner:

$$\epsilon_2 = \frac{v}{L_2}$$

Applying the constitutive relation:  $\sigma_I = E_I \epsilon_I$  we can obtain the stresses in terms of the tip displacement  $v$ :

$$\sigma_1 = E_1 \frac{L_2}{L_1^2}v, \quad \sigma_3 = E_3 \frac{L_2}{L_1^2}v, \quad \sigma_2 = E_2 \frac{v}{L_2}$$

This expressions for the strains above also apply for the case of a virtual displacement field whose value at the tip is  $\bar{v}$ . The resulting virtual strains are:

$$\bar{\epsilon}_1 = \frac{L_2}{L_1^2}\bar{v}, \quad \bar{\epsilon}_3 = \frac{L_2}{L_1^2}\bar{v}, \quad \bar{\epsilon}_2 = \frac{\bar{v}}{L_2}$$

Replacing in PVD:

$$P\bar{v} = \underbrace{A_1 L_1}_{\text{}} \underbrace{E_1 \frac{L_2}{L_1^2} v}_{\text{}} \underbrace{\frac{L_2}{L_1^2} \bar{v}}_{\text{}} + \underbrace{A_2 L_2}_{\text{}} \underbrace{E_2 \frac{v}{L_2}}_{\text{}} \underbrace{\frac{\bar{v}}{L_2}}_{\text{}} + \underbrace{A_3 L_3}_{\text{}} \underbrace{E_3 \frac{L_2}{L_1^2} v}_{\text{}} \underbrace{\frac{L_2}{L_1^2} \bar{v}}_{\text{}}$$

As expected the  $\bar{v}$ 's cancel out, as the principle must hold for all its admissible virtual values and we obtain an expression of the external load  $P$  and the resulting real displacement  $v$ . This expression can be simplified using:  $L_2 = L_1 \cos \theta = L_3 \cos \theta$ :

$$P = \frac{A_1 E_1 L_1^2 \cos^2 \theta}{L_1^3} v + A_2 E_2 \frac{v}{L_2} + \frac{A_3 E_3 L_3^2 \cos^2 \theta}{L_3^3} v$$

$$P = \left( \frac{A_1 E_1 \cos^2 \theta}{L_1} + \frac{A_2 E_2}{L_2} + \frac{A_3 E_3 \cos^2 \theta}{L_3} \right) v$$

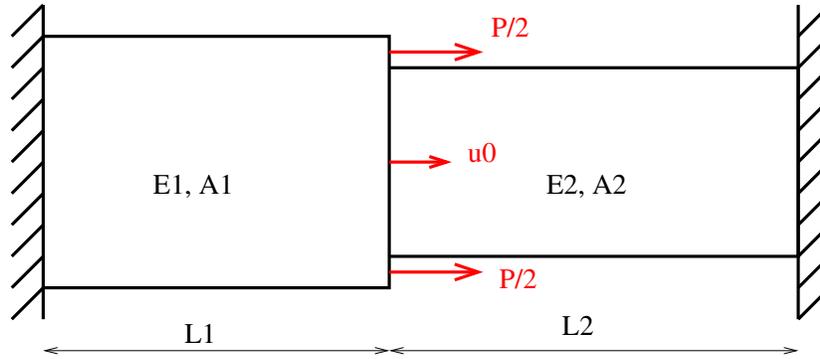
$$P = [(A_1 E_1 + A_3 E_3) \cos^3 \theta + A_2 E_2] \frac{v}{L_2}$$

$$v = \frac{P L_2}{(A_1 E_1 + A_3 E_3) \cos^3 \theta + A_2 E_2}$$

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Example:



PVD:

$$P\bar{u}_0 = A_1 L_1 \sigma_1 \bar{\epsilon}_1 + A_2 L_2 \sigma_2 \bar{\epsilon}_2$$
$$\epsilon_1 = \frac{u_0}{L_1}, \sigma_1 = E_1 \epsilon_1, \bar{\epsilon}_1 = \frac{\bar{u}_0}{L_1}$$
$$\epsilon_2 = -\frac{u_0}{L_2}, \sigma_2 = E_2 \epsilon_2, \bar{\epsilon}_2 = -\frac{\bar{u}_0}{L_2}$$
$$P\bar{u}_0 = A_1 L_1 E_1 \frac{u_0}{L_1} \frac{\bar{u}_0}{L_1} + A_2 L_2 E_2 \frac{(-u_0)}{L_2} \frac{(-\bar{u}_0)}{L_2}$$
$$P = \left( \frac{A_1 E_1}{L_1} + \frac{A_2 E_2}{L_2} \right) u_0$$

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