16.21 Techniques of Structural Analysis and Design
Spring 2004
Unit #7 (continued)- Concepts of work and energy
Strain energy and potential energy of a beam

Raúl Radovitzky

March 2, 2005

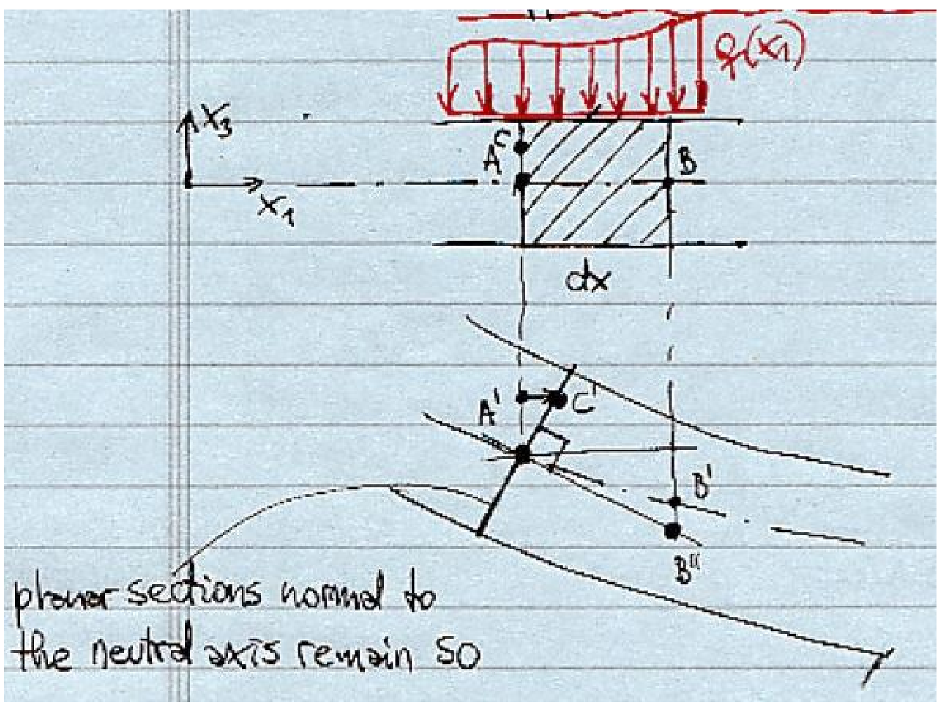


Figure 1: Kinematic assumptions for a beam

Kinematic assumptions for a beam: From the figure: $\bar{AA'} = u_3(x_1)$. Assume small deflections: $B' \sim B'', \bar{BB''} = u_3 + du_3$. $\bar{CC'} = u_3(x) +$

$u1(x_1, x_3)$. Assume *planar sections normal to the neutral axis remain planar after deformation.* Then:

$$u_3 = u_3(x_1) \tag{1}$$

$$u_1(x_1, x_3) = -x_3 \frac{du_3}{dx_1} \tag{2}$$

$$u_3(x_1) \text{ is the only primary unknown of the problem} \tag{3}$$

From these kinematic assumptions we can derive a theory for beams.

Strains:

$$\boxed{\epsilon_{11} = \frac{du_1}{dx_1} = -x_3 \frac{d^2u_3}{dx_1^2}} \tag{4}$$

$$\epsilon_{22} = \epsilon_{33} = -\nu\epsilon_{11}, \text{ plane stress} \tag{5}$$

$$\epsilon_{13} = \frac{1}{2}(\frac{du_1}{dx_3} + \frac{du_3}{dx_1}) = \frac{1}{2}(-\frac{du_3}{dx_1} + \frac{du_3}{dx_1}) = 0 \tag{6}$$

Constitutive:

$$\sigma_{11} = E\epsilon_{11} = -Ex_3 \frac{d^2u_3}{dx_1^2} \tag{7}$$

Equilibrium: Apply equilibrium (in the undeformed configuration) to *integral quantities* (moment M and shear force V). Definitions of integral quantities as forces "equivalent" to the internal stresses:

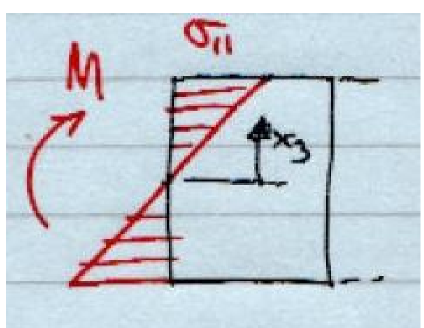
$$V(x_1) + \int_{A(x_1)} \sigma_{13} d_A = 0 \tag{8}$$

$$M(x_1) + \int_{A(x_1)} \sigma_{11} x_3 dA = 0 \tag{9}$$

replacing σ_{11}:

$$M(x_1) = \int_{A(x_1)} (-E\frac{d^2u_3}{dx_1^2} x_3^2) dA = E\frac{d^2u_3}{dx_1^2} \underbrace{\int_{A(x_1)} x_3^2 dA} \tag{10}$$

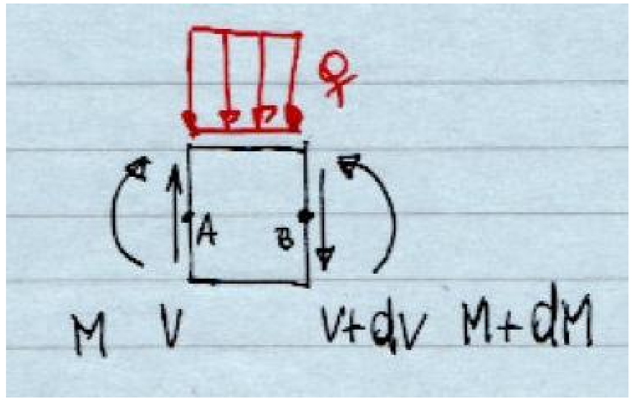
$$\boxed{M(x_1) = EI(x_1)\frac{d^2u_3}{dx_1^2}} \tag{11}$$



Also note:

$$\boxed{\sigma_{11} = -\frac{Mx_3}{I}} \tag{12}$$

With these definitions we can apply equilibrium as shown in the figure:



$$\sum F_{x_3} = 0: \quad V - q dx_1 - V - dV = 0 \;\rightarrow \boxed{\frac{dV}{dx_1} = -q} \tag{13}$$

$$\sum M^B = 0: -M + M + dM - V dx_1 + q\frac{dx_1^2}{2} = 0 \;\rightarrow \boxed{\frac{dM}{dx_1} = V} \tag{14}$$

$$\frac{d}{dx_1}\left(\frac{dM}{dx_1}\right) = -q \;\rightarrow \boxed{\frac{d^2M}{dx_1^2} = -q} \tag{15}$$

Replacing equation (11) in the last expression:

$$\boxed{\frac{d^2}{dx_1^2}\left(EI\frac{d^2u_3}{dx_1^2}\right) + q(x_1) = 0} \tag{16}$$

Fourth order differential equation governing the deflections of beams. Needs 4 boundary conditions. Examples:

- case a $u_3(0) = 0,\; u_3'(0) = 0,\; u_3''(L) = 0,\; u_3'''(L) = 0.$
- case a $u_3(0) = 0,\; u_3''(0) = 0,\; u_3(L) = 0,\; u_3''(L) = 0.$

Strain energy of a beam Start from the general definition of strain energy density:

$$U_0 = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij} \tag{17}$$

for a linear elastic material we concluded:

$$U_0 = \frac{1}{2}\sigma_{ij}\epsilon_{ij} \tag{18}$$

Classical beam theory: $\sigma_{11} \neq 0$, all other stress components are zero.

$$U_0 = \frac{1}{2}\sigma_{11}\epsilon_{11} = \frac{1}{2}E\epsilon_{11}^2 \tag{19}$$

$$\epsilon_{11} = -x_3\frac{d^2u_3}{dx_1^2} \rightarrow U_0 = \frac{1}{2}Ex_3^2\Big(\frac{d^2u_3}{dx_1^2}\Big)^2 \tag{20}$$

$$\tag{21}$$

$$U = \int_V U_0 dV = \int_V \frac{1}{2}Ex_3^2\Big(\frac{d^2u_3}{dx_1^2}\Big)^2 dV \tag{22}$$

$$= \frac{1}{2}\int_0^L E\Big(\frac{d^2u_3}{dx_1^2}\Big)^2 \int_{A(x)} x_3^2 dA dx_1 \tag{23}$$

$$\boxed{U = \frac{1}{2}\int_0^L EI(x)\Big(\frac{d^2u_3}{dx_1^2}\Big)^2 dx_1} \tag{24}$$

$$\text{also note:} \tag{25}$$

$$\boxed{U = \frac{1}{2}\int_0^L M(x_1)\frac{d^2u_3}{dx_1^2} dx_1} \tag{26}$$

Complementary strain energy of a beam

Complementary strain energy density:

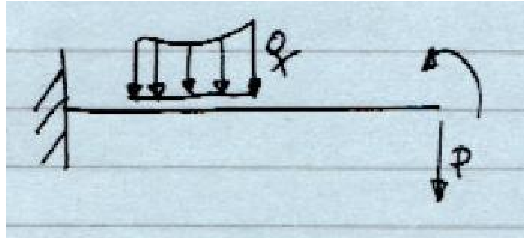
$$U_0^* = \int_0^{\epsilon_{11}} \epsilon_{11} d\sigma_{11} = \frac{1}{2}\frac{\sigma_{11}^2}{E} = \frac{1}{2E}\Big(\frac{-Mx_3}{I}\Big)^2 \tag{27}$$

The complementary strain energy is then

$$U_c = \int_V U_0^* dV = \frac{1}{2}\int_0^L \frac{M^2}{EI^2}\int_{A(x_1)} x_3^2 dA dx_1 \tag{28}$$

$$\boxed{U_c = \frac{1}{2}\int_0^L \frac{M^2}{EI} dx_1} \tag{29}$$

Potential of the external forces:



$$V = -\int_S t_i u_i dS - \int_V f_i u_i dV \tag{30}$$

$$\boxed{V = -\int_0^L q(x_1)u_3(x_1)dx_1 - Pu_3(L) - Mu_3'(L)} \tag{31}$$

The total potential energy of the beam is:

$$\boxed{\Pi(u_3) = \int_0^L \left[\frac{1}{2}EI\left(\frac{d^2u_3}{dx^2}\right)^2 dx_1 + qu_3\right] + Pu_3(L) + Mu_3'(L)} \tag{32}$$

This is the expression we gave the first day of class!.