16.21 Techniques of Structural Analysis and Design
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Unit #4 - Thermodynamics Principles

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First Law of Thermodynamics

$$\boxed{\frac{d}{dt}\big(K+U\big) = P + H} \tag{1}$$

where:

- K: kinetic energy
- U: internal energy
- P: Power of external forces
- H: hear exchange per unit time

$$K = \frac{1}{2}\int_V \rho \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial \mathbf{u}}{\partial t} dV = \frac{1}{2}\int_V \rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} dV \tag{2}$$

$$U = \int_V \rho \widehat{U}_0 dV = \int_V U_0 dV \tag{3}$$

where $\widehat{U}_0, U_0$ are the internal energy densities per unit mass and per unit volume, respectively.

$$P = \int_V \mathbf{f} \cdot \frac{\partial \mathbf{u}}{\partial t} dV + \int_S \mathbf{t} \cdot \frac{\partial \mathbf{u}}{\partial t} dS \tag{4}$$

In components:

$$P = \int_V f_i \frac{\partial u_i}{\partial t} dV + \int_S t_i \frac{\partial u_i}{\partial t} dS \tag{5}$$

Replacing $t_i = n_j \sigma ji$ in this expression:

$$P = \int_V f_i \frac{\partial u_i}{\partial t} dV + \int_S n_j \sigma ji \frac{\partial u_i}{\partial t} dS \tag{6}$$

Using Gauss' Theorem:

$$\begin{aligned} P &= \int_V f_i \frac{\partial u_i}{\partial t} dV + \int_V \frac{\partial}{\partial x_j} \Big(\sigma_{ji} \frac{\partial u_i}{\partial t} \Big) dV \\ &= \int_V \Big[\underbrace{\Big(\frac{\partial \sigma_{ji}}{\partial x_j} + f_i \Big)}_{\rho \frac{\partial^2 u_i}{\partial t^2} \text{ (why?)}} \frac{\partial u_i}{\partial t} + \sigma_{ji} \underbrace{\frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t}}_{\frac{\partial}{\partial t} \frac{\partial u_i}{\partial x_j}} \Big] dV \\ &= \int_V \Big(\underbrace{\rho \frac{\partial^2 u_i}{\partial t^2} \frac{\partial u_i}{\partial t}}_{\frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial u_i}{\partial t} \right)^2} + \underbrace{\sigma_{ji} \frac{\partial}{\partial t} \frac{\partial u_i}{\partial x_j}}_{\sigma_{ji} \frac{\partial}{\partial t} \epsilon_{ji}} \Big) dV \end{aligned} \tag{7}$$

Notation:
Time derivatives:

$$\boxed{\frac{\partial (\ \ )}{\partial t} = (\dot{\ \ })}$$

Examples:

- $\frac{\partial u_i}{\partial t} = \dot{u}_i, \ \frac{\partial \mathbf{u}}{\partial t} = \dot{\mathbf{u}}$
- $\frac{\partial^2 u_i}{\partial t^2} = \ddot{u}_i$

- $\frac{\partial \epsilon_{ij}}{\partial t} = \dot{\epsilon}_{ij}$

Spatial derivatives:

$$\boxed{\frac{\partial (\quad)}{\partial x_i} = (\quad)_{,i}}$$

Examples:

- $\frac{\partial \sigma_{ji}}{\partial x_j} = \sigma_{ji,j}$

With this notation, the power of the external forces can be rewritten as:

$$P = \underbrace{\frac{d}{dt}\int_V \frac{1}{2}\rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} dV}_{K} + \underbrace{\int_V \sigma_{ji}\dot{\epsilon}_{ji} dV}_{\text{deformation power}} \tag{8}$$

where the "ρdV" inside the first integral was included inside the time derivative since it is a constant due to conservation of mass. We conclude that part of the power of the external forces goes into changing the kinetic energy of the material and the rest into deforming the material. We call the latter the *deformation power* and it represents the rate at which the stresses do work on the deforming material.

Replacing in the first law, equation (1):

$$\frac{d}{dt}\big(K + U\big) = \frac{d}{dt}\big(K\big) + \int_V \sigma_{ji}\dot{\epsilon}_{ji} dV + H \tag{9}$$

After canceling the kinetic energy from both sides, the first law expresses the fact that the internal energy of a deforming material can be changed either by heating or by deforming the material:

$$\boxed{\frac{dU}{dt} = \frac{d}{dt}\int_V \rho \widehat{U}_0 dV = \int_V \sigma_{ji}\dot{\epsilon}_{ji} dV + H} \tag{10}$$

In the isothermal case ($H = 0$):

$$\int_V \Big(\rho \frac{\partial \widehat{U}_0}{\partial t} - \sigma_{ij}\dot{\epsilon}_{ij}\Big) dV = 0 \tag{11}$$

or, in local form:

$$\rho \frac{\partial \widehat{U}_0}{\partial t} = \sigma_{ij} \dot{\epsilon}_{ij} \tag{12}$$

In ideal elasticity, we assume that all the work of deformation is converted into internal energy, i.e., the internal energy density is a *state function* of the deformation:

$$\widehat{U}_0 = \widehat{U}_0(\epsilon_{ij}) \tag{13}$$

Then:

$$\frac{\partial \widehat{U}_0}{\partial t} = \frac{\partial \widehat{U}_0}{\partial \epsilon_{ij}} \dot{\epsilon}_{ij} \tag{14}$$

Replace in first law, equation (12:

$$\rho \frac{\partial \widehat{U}_0}{\partial \epsilon_{ij}} \dot{\epsilon}_{ij} = \sigma_{ij} \dot{\epsilon}_{ij} \ \Rightarrow \tag{15}$$

$$\boxed{\rho \frac{\partial \widehat{U}_0}{\partial \epsilon_{ij}} = \sigma_{ij}} \tag{16}$$

i.e., the stresses derive from a potential.