16.21 Techniques of Structural Analysis and Design
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Unit #3 - Kinematics of deformation

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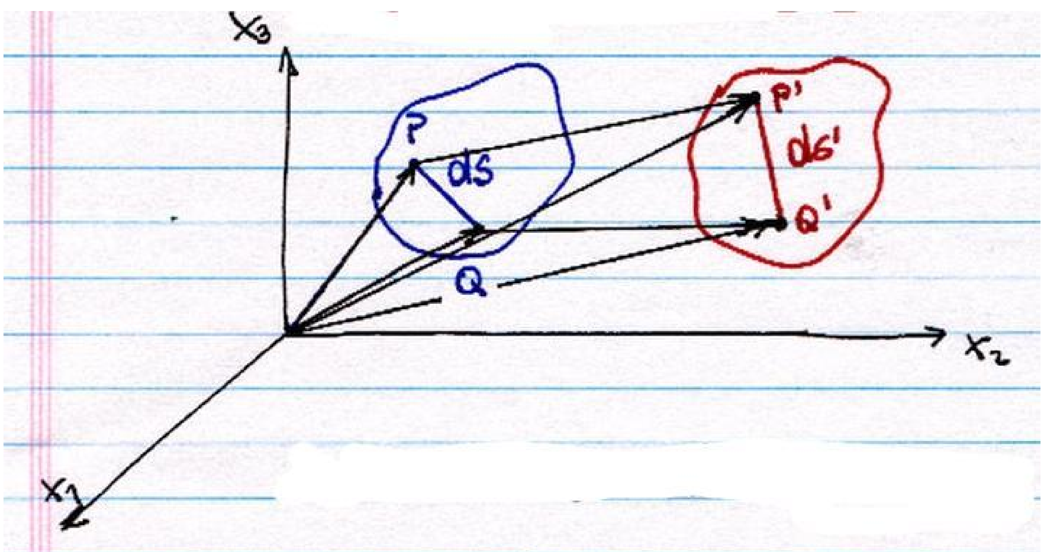

Figure 1: Kinematics of deformable bodies

Deformation described by *deformation mapping*:

$$\mathbf{x}' = \varphi(\mathbf{x}) = \mathbf{x} + \mathbf{u} \tag{1}$$

We seek to characterize the local state of deformation of the material in a neighborhood of a point P. Consider two points P and Q in the undeformed:

$$P : \mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i \tag{2}$$

$$Q : \mathbf{x} + \mathbf{dx} = (x_i + dx_i)\mathbf{e}_i \tag{3}$$

and deformed

$$P' : \mathbf{x}' = \varphi_1(\mathbf{x})\mathbf{e}_1 + \varphi_2(\mathbf{x})\mathbf{e}_2 + \varphi_3(\mathbf{x})\mathbf{e}_3 = \varphi_i(\mathbf{x})\mathbf{e}_i \tag{4}$$

$$Q' : \mathbf{x}' + \mathbf{dx}' = \big(\varphi_i(\mathbf{x}) + d\varphi_i\big)\mathbf{e}_i \tag{5}$$

configurations. In this expression,

$$\mathbf{dx}' = d\varphi_i\mathbf{e}_i \tag{6}$$

Expressing the differentials $d\varphi_i$ in terms of the partial derivatives of the functions $\varphi_i(x_j\mathbf{e}_j)$:

$$d\varphi_1 = \frac{\partial\varphi_1}{\partial x_1}dx_1 + \frac{\partial\varphi_1}{\partial x_2}dx_2 + \frac{\partial\varphi_1}{\partial x_3}dx_3, \tag{7}$$

and similarly for $d\varphi_2, d\varphi_3$, in index notation:

$$d\varphi_i = \frac{\partial\varphi_i}{\partial x_j}dx_j \tag{8}$$

Replacing in equation (5):

$$Q' : \mathbf{x}' + \mathbf{dx}' = \Big(\varphi_i + \frac{\partial\varphi_i}{\partial x_j}dx_j\Big)\mathbf{e}_i \tag{9}$$

$$\mathbf{dx}'_i = \frac{\partial\varphi_i}{\partial x_j}dx_j\mathbf{e}_i \tag{10}$$

We now try to compute the change in length of the segment $\overrightarrow{PQ}$ which deformed into segment $\overrightarrow{P'Q'}$. Undeformed length (to the square):

$$ds^2 = \|\mathbf{dx}\|^2 = \mathbf{dx}\cdot\mathbf{dx} = dx_idx_i \tag{11}$$

Deformed length (to the square):

$$(ds')^2 = \|\mathbf{dx}'\|^2 = \mathbf{dx}' \cdot \mathbf{dx}' = \frac{\partial \varphi_i}{\partial x_j} dx_j \frac{\partial \varphi_i}{\partial x_k} dx_k \tag{12}$$

The change in length of segment $\overrightarrow{PQ}$ is then given by the difference between equations (12) and (11):

$$(ds')^2 - ds^2 = \frac{\partial \varphi_i}{\partial x_j} dx_j \frac{\partial \varphi_i}{\partial x_k} dx_k - dx_i dx_i \tag{13}$$

We want to extract as common factor the differentials. To this end we observe that:

$$dx_i dx_i = dx_j dx_k \delta_{jk} \tag{14}$$

Then:

$$\begin{aligned} (ds')^2 - ds^2 &= \frac{\partial \varphi_i}{\partial x_j} dx_j \frac{\partial \varphi_i}{\partial x_k} dx_k - dx_j dx_k \delta_{jk} \\ &= \underbrace{\left(\frac{\partial \varphi_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_k} - \delta_{jk} \right)}_{2\epsilon_{jk}:\ \textit{Green-Lagrange strain tensor}} dx_j dx_k \end{aligned} \tag{15}$$

Assume that the deformation mapping $\varphi(\mathbf{x})$ has the form:

$$\varphi(\mathbf{x}) = \mathbf{x} + \mathbf{u} \tag{16}$$

where $\mathbf{u}$ is the *displacement field.* Then,

$$\frac{\partial \varphi_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} = \delta_{ij} + \frac{\partial u_i}{\partial x_j} \tag{17}$$

and the Green-Lagrange strain tensor becomes:

$$\begin{aligned} 2\epsilon_{ij} &= \left(\delta_{mi} + \frac{\partial u_m}{\partial x_i} \right) \left(\delta_{mj} + \frac{\partial u_m}{\partial x_j} \right) - \delta_{ij} \\ &= \cancel{\delta_{ij}} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} - \cancel{\delta_{ij}} \end{aligned} \tag{18}$$

$$\text{Green-Lagrange strain tensor : } \boxed{\epsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\Big)} \tag{19}$$

When the absolute values of the derivatives of the displacement field are much smaller than 1, their products (nonlinear part of the strain) are even smaller and we'll neglect them. We will make this assumption throughout this course (See accompanying Mathematica notebook evaluating the limits of this assumption). Mathematically:

$$\Big\|\frac{\partial u_i}{\partial x_j}\Big\| \ll 1 \ \Rightarrow \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j} \sim 0 \tag{20}$$

We will define the *linear part* of the Green-Lagrange strain tensor as the *small strain tensor*:

$$\boxed{\epsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)} \tag{21}$$

Transformation of strain components

Given: $\epsilon_{ij}, \mathbf{e}_i$ and a new basis $\tilde{\mathbf{e}}_k$, determine the components of strain in the new basis $\tilde{\epsilon}_{kl}$

$$\tilde{\epsilon}_{ij} = \frac{1}{2}\Big(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i}\Big) \tag{22}$$

We want to express the expressions with tilde on the right-hand side with their non-tilde counterparts. Start by applying the chain rule of differentiation:

$$\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = \frac{\partial \tilde{u}_i}{\partial x_k}\frac{\partial x_k}{\partial \tilde{x}_j} \tag{23}$$

Transform the displacement components:

$$\mathbf{u} = \tilde{u}_m \tilde{\mathbf{e}}_m = u_l \mathbf{e}_l \tag{24}$$

$$\tilde{u}_m(\tilde{\mathbf{e}}_m \cdot \tilde{\mathbf{e}}_i) = u_l(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \tag{25}$$

$$\tilde{u}_m \delta_{mi} = u_l(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \tag{26}$$

$$\tilde{u}_i = u_l(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \tag{27}$$

take the derivative of $\tilde{u}_i$ with respect to x_k, as required by equation (23):

$$\frac{\partial \tilde{u}_i}{\partial x_k} = \frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \tag{28}$$

and take the derivative of the reverse transformation of the components of the position vector $\mathbf{x}$:

$$\mathbf{x} = x_j \mathbf{e}_j = \tilde{x}_k \tilde{\mathbf{e}}_k \tag{29}$$

$$x_j(\mathbf{e}_j \cdot \mathbf{e}_i) = \tilde{x}_k(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) \tag{30}$$

$$x_j \delta_{ji} = \tilde{x}_k(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) \tag{31}$$

$$x_i = \tilde{x}_k(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) \tag{32}$$

$$\frac{\partial x_i}{\partial \tilde{x}_j} = \frac{\partial \tilde{x}_k}{\partial \tilde{x}_j}(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) = \delta_{kj}(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) = (\tilde{\mathbf{e}}_j \cdot \mathbf{e}_i) \tag{33}$$

Replacing equations (28) and (33) in (23):

$$\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = \frac{\partial \tilde{u}_i}{\partial x_k}\frac{\partial x_k}{\partial \tilde{x}_j} = \frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) \tag{34}$$

Replacing in equation (22):

$$\tilde{\epsilon}_{ij} = \frac{1}{2}\left[\frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) + \frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_j)(\tilde{\mathbf{e}}_i \cdot \mathbf{e}_k)\right] \tag{35}$$

Exchange indices l and k in second term:

$$\begin{aligned} \tilde{\epsilon}_{ij} &= \frac{1}{2}\left[\frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) + \frac{\partial u_k}{\partial x_l}(\mathbf{e}_k \cdot \tilde{\mathbf{e}}_j)(\tilde{\mathbf{e}}_i \cdot \mathbf{e}_l)\right] \\ &= \frac{1}{2}\left(\frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l}\right)(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) \end{aligned} \tag{36}$$

Or, finally:

$$\boxed{\tilde{\epsilon}_{ij} = \epsilon_{lk}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k)} \tag{37}$$

Compatibility of strains

Given displacement field $\mathbf{u}$, expression (21) allows to compute the strains components ϵ_{ij}. How does one answer the reverse question? Note analogy with potential-gradient field. Restrict the analysis to two dimensions:

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad 2\epsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \tag{38}$$

Differentiate the strain components as follows:

$$\frac{\partial^2}{\partial x_1 \partial x_2}\big(2\epsilon_{12}\big) = \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2} \tag{39}$$

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} = \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} \tag{40}$$

$$\frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = \frac{\partial^3 u_2}{\partial x_2 \partial x_1^2} \tag{41}$$

and conclude that:

$$\boxed{2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2}} \tag{42}$$