16.21 Techniques of Structural Analysis and Design
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Unit #1

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In this course we are going to focus on *energy and variational methods* for structural analysis. To understand the overall approach we start by contrasting it with the alternative *vector mechanics* approach:

Example of Vector mechanics formulation:

Consider a simply supported beam subjected to a uniformly-distributed load q_0, (see Fig.1). To analyze the equilibrium of the beam we consider the free body diagram of an element of length Δx as shown in the figure and apply Newton's second law:

$$\sum F_y = 0 \,:\, V - q_0 \Delta x - (V + \Delta V) = 0 \tag{1}$$

$$\sum M_B = 0 \,:\, -V\Delta x - M + (M + \Delta M) + (q_0 \Delta x)\frac{\Delta x}{2} = 0 \tag{2}$$

Dividing by Δx and taking the limit $\Delta x \to 0$:

$$\frac{dV}{dx} = -q_0 \tag{3}$$

$$\frac{dM}{dx} = V \tag{4}$$

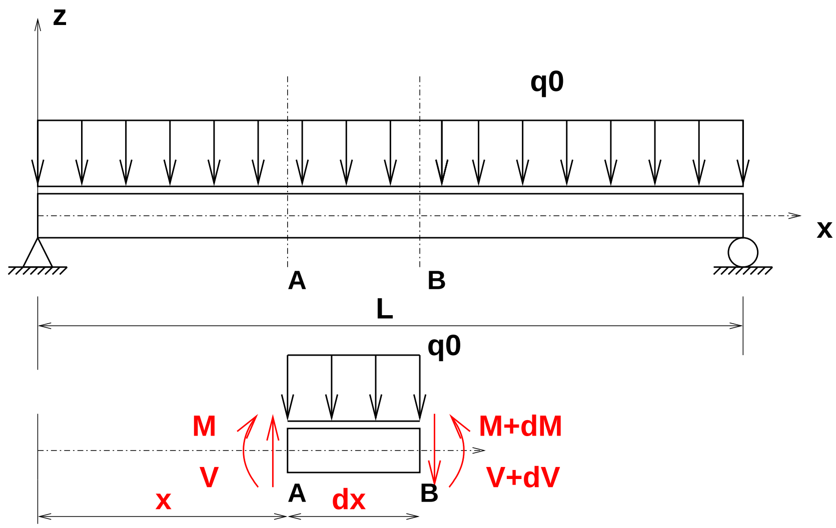

Figure 1: Equilibrium of a simply supported beam

Eliminating V, we obtain:

$$\frac{d^2M}{dx^2} + q_0 = 0 \tag{5}$$

Recall from Unified Engineering or 16.20 (we'll cover this later in the course also) that the bending moment is related to the deflection of the beam $w(x)$ by the equation:

$$M = EI\frac{d^2w}{dx^2} \tag{6}$$

where E is the Young's modulus and I is the moment of inertia of the beam. Combining 5 and 6, we obtain:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) + q_0 = 0, \quad 0 < x < L \tag{7}$$

The boundary conditions of the beam are:

$$w(0) = w(L) = 0, \quad M(0) = M(L) = 0 \tag{8}$$

The solution of equations 7 and 8 is given by:

$$w(x) = -\frac{q_0}{24EI}x\left(L - x\right)\left(L^2 + Lx - x^2\right) \tag{9}$$

Corresponding variational formulation

The same problem can be formulated in variational form by introducing the potential energy of the beam system:

$$\Pi(w) = \int_0^L \left[\frac{EI}{2} \left(\frac{d^2 w}{dx^2} \right)^2 + q_0 w \right] dx \tag{10}$$

and requiring that the solution $w(x)$ be the function minimizing it that also satisfies the displacement boundary conditions:

$$w(0) = w(L) = 0 \tag{11}$$

A particularly attractive use of the variational formulation lies in the determination of approximate solutions. Let's seek an approximate solution to the previous beam example of the form:

$$w_1(x) = c_1 x(L - x) \tag{12}$$

which has a continuous second derivative and satisfies the boundary conditions 11. Substituting $w_1(x)$ in 10 we obtain:

$$\begin{aligned} \Pi(c_1) &= \int_0^L \left[\frac{EI}{2} (-2c_1)^2 + q_0 c_1 (Lx - x^2) \right] dx \\ &= 2EILc_1^2 + \frac{L^3}{6} q_0 c_1 \end{aligned} \tag{13}$$

Note that our functional Π now depends on c_1 only. $w_1(x)$ is an approximate solution to our problem if c_1 minimizes $\Pi = \Pi(c_1)$. A necessary condition for this is:

$$\frac{d\Pi}{dc_1} = 4EILc_1 + q_0 \frac{L^3}{6} = 0$$

or $c_1 = -\frac{q_0 L^2}{24EI}$, and the approximate solution becomes:

$$w_1(x) = -\frac{q_0 L^2}{24EI} x(L - x)$$

In order to assess the accuracy of our approximate solution, let's compute the approximate deflection of the beam at the midpoint $\delta_1 = w_1(\frac{L}{2})$:

$$\delta = -\frac{q_0 L^2}{24EI} \left(\frac{L}{2} \right)^2 = -\frac{q_0 L^4}{96EI}$$

The exact value $\delta = w(\frac{L}{2})$ is obtained from eqn.9 as:

$$\delta = -\frac{q_0}{24EI}\frac{L}{2}\left(L - \frac{L}{2}\right)\left[L^2 + L\frac{L}{2} - \left(\frac{L}{2}\right)^2\right] = -\frac{5}{384}\frac{q_0 L^4}{EI}$$

We observe that:

$$\frac{\delta_1}{\delta} = \frac{\frac{1}{96}}{\frac{5}{384}} = \frac{4}{5} = 0.8$$

i.e. the approximate solution underpredicts the maximum deflection by 20%.

However, if we consider the following approximation with 3 degrees of freedom (note it also satisfies the essential boundary conditions, eqn.11):

$$w_3(x) = c_1 x(L - x) + c_2 x^2(L - x) + c_3 x^3(L - x) \tag{14}$$

and require that $\Pi(c_1, c_2, c_3)$ be a minimum:

$$\frac{\partial \Pi}{\partial c_1} = 0, \frac{\partial \Pi}{\partial c_2} = 0, \frac{\partial \Pi}{\partial c_3} = 0,$$

i.e.:

$$4\,c1\,EI\,L + 2\,c2\,EI\,L^2 + 2\,c3\,EI\,L^3 + \frac{L^3\,q0}{6} = 0$$

$$2\,c1\,EI\,L^2 + 4\,c2\,EI\,L^3 + 4\,c3\,EI\,L^4 + \frac{L^4\,q0}{12} = 0$$

$$2\,c1\,EI\,L^3 + 4\,c2\,EI\,L^4 + \frac{24\,c3\,EI\,L^5}{5} + \frac{L^5\,q0}{20} = 0$$

whose solution is:

$$\boxed{c1 \to \frac{-\left(L^2\,q0\right)}{24\,EI}, c2 \to \frac{-\left(L\,q0\right)}{24\,EI}, c3 \to \frac{q0}{24\,EI}}$$

If you replace this values in eqn. 14 and evaluate the deflection at the midpoint of the beam you obtain the exact solution !!!