Numerical integration

Consider the 1-D integral:

$$\boxed{I(f) = \int_{-1}^{1} f(\xi)\, d\xi}$$

Seek n-point approximations:

$$\boxed{I(f) \sim \sum_{q=1}^{m} w_q f(\xi_q) = I_q(f)}$$

where W_q are the weights and
ξ_q are the Gauss (sampling) points

> Gauss quadrature: select the "m" sampling points and weights so that the rule is exact for the polynomial of highest order possible

• One-point formula ($m=1$)

$I_q(f) = w_1 f(\xi_1)$, we have one weight (w_1)

and one sampling point (ξ_1) to determine. We should be able to integrate exactly a polynomial with two parameters, i.e., a linear function: $f = a_0 + a_1 \xi$

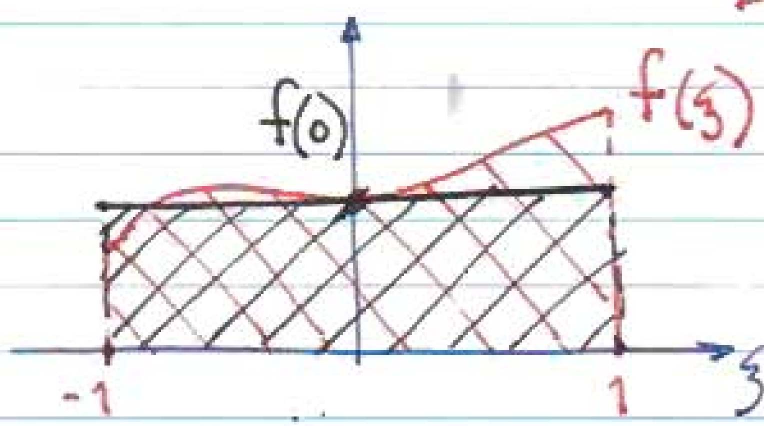
$$I(f) = \int_{-1}^{1} (a_0 + a_1 \xi)\, d\xi = 2a_0$$

Setting $I_q(f) = I(f)$, we obtain values for the parameters:

$$\boxed{2a_0 = w_1 (a_0 + a_1 \xi_1)}$$

This is satisfied if: $\boxed{\xi_1 = 0, \quad w_1 = 2}$

which gives the "midpoint rule" $\boxed{I_1(f) = 2f(0)}$



- **Two-point formula** ($m=2$)

$$I_2(f) = w_1 f(\xi_1) + w_2 f(\xi_2) \;, \quad \text{2 Gauss points, 2 weights}$$

Polynomial with 4 parameters:

$$f = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 \qquad \text{(cubic)}$$

Exact integral:

$$I(f) = \int_{-1}^{1} \left(a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3\right) d\xi$$

$$= 2a_0 + \frac{2}{3} a_2$$

Approximate integral:

$$I_2(f) = (W_1 + W_2)\, a_0 + (W_1 \xi_1 + W_2 \xi_2)\, a_1 +$$
$$+ (W_1 \xi_1^2 + W_2 \xi_2^2)\, a_2 + (W_1 \xi_1^3 + W_2 \xi_1^3)\, a_3$$

$$\Rightarrow \begin{cases} W_1 + W_2 = 2 \\ W_1 \xi_1 + W_2 \xi_2 = 0 \\ W_1 \xi_1^2 + W_2 \xi_2^2 = 2/3 \\ W_1 \xi_1^3 + W_2 \xi_2^3 = 0 \end{cases} \longrightarrow \boxed{\begin{array}{c} W_1 = W_2 = 1 \\ \hline \xi_{1,2} = \mp \dfrac{1}{\sqrt{3}} \end{array}}$$

$$\Longrightarrow \boxed{I_2(f) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)}$$

Examples: $f(\xi) = \cos(\xi)$

Exact: $I = \int_{-1}^{1} \cos\xi \, d\xi = -\sin\xi \Big|_{-1}^{1} = 2\sin 1 = 1.68$

$$I_1 = 2\cos(0) = 2$$

$$I_2 = \cos\left(\frac{-1}{\sqrt{3}}\right) + \cos\left(-\frac{1}{\sqrt{3}}\right) = 2\cos\left(\frac{1}{\sqrt{3}}\right) = 1.676$$

Two-dimensional Integrals

$$I(f) = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi\, d\eta$$

$$\sim \int_{-1}^{1} \sum_{p=1}^{m} w_p f(\xi_p,\eta)\, d\eta$$

$$\sim \boxed{\sum_{q=1}^{m} \sum_{p=1}^{m} w_p w_q f(\xi_p,\eta_q) = I_{p,q} = I_m}$$