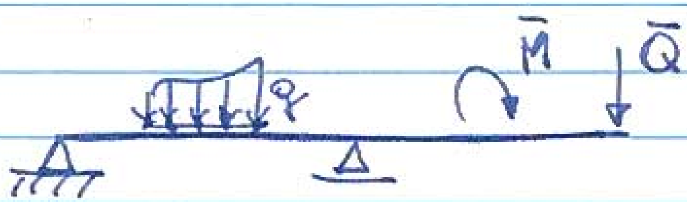
Finite element model of a beam (Euler-Bernoulli)

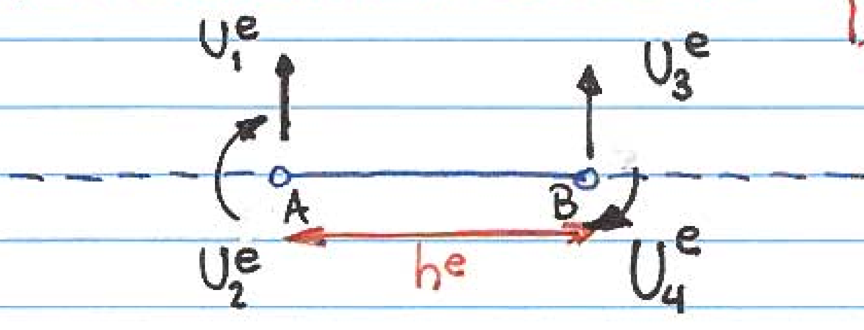
Governing equations:

$$
\begin{cases}
(EI\, w'')'' - q = 0 & 0 < x < L \\
\left.\begin{aligned} w &= \bar{w} \\ w' &= \bar{w}' \end{aligned}\right\} \text{ on } S_u & \text{(displacement BC)} \\
\left.\begin{aligned} EI\, w'' &= \bar{M} \\ (EI\, w'')' &= \bar{Q} \end{aligned}\right\} \text{ on } S_t & \text{(natural or traction BC)}
\end{cases}
$$



Approximation inside element

$$\boxed{w_0^e(x) \approx \sum_{i=1}^{n} \phi_i^e U_i^e}$$



"Element boundary conditions"

- displacement $w(x_1^e) = U_1^e \qquad w'(x_1^e) = U_2^e \qquad w(x_2^e) = U_3^e \qquad w'(x_2^e) = U_4^e$

• natural $\quad (EI\,w'')'\Big|_{x_A^e} = Q_1^e \qquad (EI\,w'')'\Big|_{x_B} = -Q_3^e$

$$EI\,w''\Big|_{x_A^e} = Q_2^e \qquad EI\,w''\Big|_{x_B^e} = -Q_4^e$$

What do the first and second row represent?

Potential energy for the beam element

$$\boxed{\Pi^e(w_0^e) = \int_{x_A}^{x_B}\left[E_e I_e \left(\frac{d^2 w_0^e}{dx^2}\right)^2 + w_0^e\, q_0^e\right]dx - P_1^e U_1^e - P_2^e U_2^e - P_3^e U_3^e - P_4^e U_4^e}$$

What do the last four terms represent?

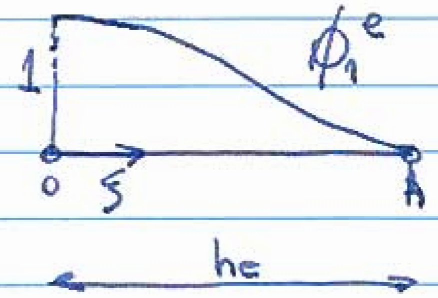
Derivation of basis function ϕ_i

Need twice-differentiable, continuous, continuous-slope functions. The minimum polynomial order should be "three", so that non-zero shears are obtained at the nodes. The cubic polynomial also gives us for parameters to fit the four essential boundary conditions at the nodes.

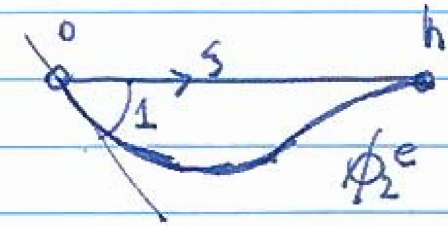
The resulting basis functions are the

Hermite cubic polynomials

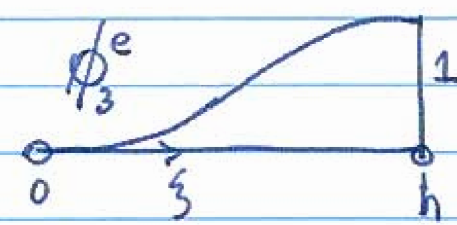
$$\phi_1^e = 1 - 3\left(\frac{\xi}{h_e}\right)^2 + 2\left(\frac{\xi}{h_e}\right)^3$$



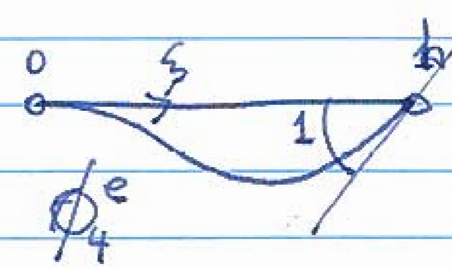
$$\phi_2^e = -\xi\left[1 - \left(\frac{\xi}{h_e}\right)\right]^2$$



$$\phi_3^e = 3\left(\frac{\xi}{h_e}\right)^2 - 2\left(\frac{\xi}{h_e}\right)^3$$



$$\phi_4^e = -\xi\left[\left(\frac{\xi}{h_e}\right)^2 - \frac{\xi}{h_e}\right]$$



Finite element equations:

Replace approximation $w_0^e = \sum_{i=1}^{4} \phi_i^e U_i^e$ into the potential Π:

$$\Pi^e(U_i^e) = \int_{x_A}^{x_B} \left[\frac{E_e I_e}{2} \left(\sum_{j=1}^{4} U_j^e \frac{d^2\phi_j^e}{dx^2} \right)^2 + \sum_{i=1}^{4} U_i^e \phi_i^e q_0^e \right] dx$$

$$- \sum_{i=1}^{4} P_i^e U_i^e$$

Apply PMPE: $\boxed{\delta \Pi^e = 0} \Leftrightarrow \boxed{\frac{\partial \Pi^e}{\partial U_i^e} = 0}$

$$\frac{\partial \Pi^e}{\partial U_i^e} = \int_{x_A}^{x_B} \left[E_e I_e \; U_j^e \frac{d^2\phi_j^e}{dx^2} \underbrace{\frac{\partial U_k^e}{\partial U_i^e}}_{\delta_{ik}} \frac{d^2\phi_k^e}{dx^2} + \right.$$

$$\left. + q_0^e \underbrace{\frac{\partial U_k^e}{\partial U_i^e}}_{\delta_{ik}} \phi_k^e \right] dx - P_k^e \underbrace{\frac{\partial U_k^e}{\partial U_i^e}}_{\delta_{ik}} = 0$$

$$0 = \underbrace{\int_{x_A}^{x_B} E_e I_e \frac{d^2\phi_i}{dx^2} \frac{d^2\phi_j}{dx^2} dx}_{K_{ij}^e} \underbrace{U_j^e}_{U_j^e} + \underbrace{\int_{x_A}^{x_B} q_0^e \phi_i^e dx - P_i^e}_{R_i^e}$$

$$K_{ij}^e \, U_j^e = R_i^e$$

element stiffness matrix — element nodal displacements — element force vector

For the case in which E_e and I_e are constant inside the element, these reduce to:

$$\boxed{K^e = \frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h \\ & 2h^2 & 3h & h^2 \\ & & 6 & 3h \\ \text{sym} & & & 2h^2 \end{bmatrix}}$$

$$\boxed{R^e = \frac{-q_0 h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix} + \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}}$$