Approximate Methods

PMPE and PVD provide alternative formulation for problems in structural mechanics. However they don't tell us how to obtain the solution, just some conditions that the solution must satisfy.
A key observation is that these principles result in algebraic (usually linear) systems of equations when the unknown solution fields are replaced with a linear combination of functions of assumed functional dependence. The unknowns of the system are the parameters appearing in the linear combination of functions. The resulting linear combination with parameters determined from the solution of the algebraic system is an approximation to the exact solution. Since the exact solution cannot in general be represented by a finite linear combination of simple functions an error is introduced.

Rayleigh-Ritz Method

Introduce a linear combination of functions of given functional form as the approximate solution of our displacement field $u_i(x_j)$

$$u_i(x_j) \approx U_i(x_j) = c^{k}_{(i)} \phi^{k}_{i}(x_j) \qquad k = 1, \ldots, n$$

We have effectively replaced our "infinite dimensional" problem of determining $u_i(x_j)$ with the problem of determining the $\boxed{3\times n}$ $\boxed{\text{coefficients}}$ or parameters $\boxed{C_i^k}$. Towards this end we introduce our approximation $\boxed{U_i(x_j)}$ in the definition of the potential energy of an elastic body replacing the exact solution $\boxed{u_i(x_k)}$

$$\boxed{\Pi(u_i(x_k)) \approx \Pi(U_i(x_k))} = \Pi(C_i^k)$$

This results in an expression whose only unknowns are the coefficients c_i^k. These are obtained by invoking the PMPE. A necessary condition for this is that ALL the derivatives of the approximate Π with respect to its parameters should vanish. This furnishes $\boxed{3\times n}$ equations:

$$\boxed{\frac{\partial \Pi}{\partial c_i^k} = 0}$$

Matrix form of the Ritz equations:

Consider the potential energy of a linear elastic material:

$$\Pi(u_i) = \frac{1}{2}\int_V \sigma_{ij}\,\varepsilon_{ij}\,dV - \int_S t_i\,u_i\,ds - \int_V f_i\,u_i\,dV$$

$$= \frac{1}{2}\int_V C_{ijkl}\,\varepsilon_{kl}\,\varepsilon_{ij}\,dV - \int_S t_i u_i\,dS - \int_V f_i u_i\,dV$$

For the approximate solution, u_i is replaced with U_i

$$\varepsilon_{ij} \approx \mathcal{E}_{ij} = \frac{1}{2}\left(U_{i,j} + U_{j,i}\right) = \frac{1}{2}\left[c^k_{(i)}\,\phi^k_{i,j} + c^k_{(j)}\,\phi^k_{j,i}\right]$$

$$\Pi(U_i) = \Pi(c^k_i) = \frac{1}{2}\int_V C_{ijkl}\,\frac{1}{2}\left(c^m_{(k)}\,\phi^m_{k,l} + c^m_{(l)}\,\phi^m_{l,k}\right)\cdot$$

$$\cdot\frac{1}{2}\left(c^n_{(i)}\,\phi^n_{i,j} + c^n_{(j)}\,\phi^n_{j,i}\right)dV - \int_V t_i\,c^k_{(i)}\,\phi^k_i\,dS - \int_V f_i\,c^k_{(i)}\,\phi^k_i\,dV$$

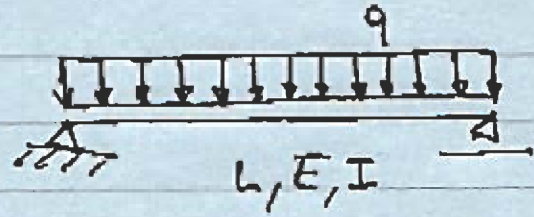
And this gets really messy when expanded. The important thing to note is that because of the quadratic dependence of the internal energy on ε_{ij}, the approximate strain energy will have terms that are quadratic in the unknown $c^k_{(i)}$'s. The potential of the external forces only has linear dependence on $c^k_{(i)}$'s. When the derivatives

$\dfrac{\partial \Pi}{\partial c^k_{(i)}}$ are computed we obtain a ~~matrix of coefficients~~ linear system on the $c^k_{(i)}$ unknowns.
involving

This corresponds to the general 3D problem. We will apply this procedure to specific problems (beams) for which things look simpler.

Example: Simply supported beam:

- uniform distributed load



$$\Pi(u_3) = \frac{1}{2}\int_0^L EI\left(\frac{d^2u_3}{dx_1^2}\right)^2 dx_1 + \int_0^L q(x_1)\, u_3(x_1)\, dx_1$$

$$\approx \Pi(U_3) = \frac{1}{2}\int_0^L EI \left[C^k \frac{d^2\phi^k}{dx_1^2}\right]^2 dx_1 + \int_0^L q(x_1)\, C^k \phi^k_{(x_1)}\, dx$$

$$\delta\Pi = 0 \implies \frac{\partial \Pi(C^k)}{\partial C^l} = 0$$

$$\frac{\partial \Pi(C^k)}{\partial C^l} = \frac{1}{\cancel{2}}\int_0^L \cancel{2}\, EI\, C^k \left(\frac{d^2\phi^k}{dx_1^2}\right) \overbrace{\frac{\partial C^k}{\partial C^l}}^{\delta_{kl}} \frac{d^2\phi^k}{dx_1^2}\, dx_1 +$$

$$+ \int_0^L q(x_1)\, \overbrace{\frac{\partial C^k}{\partial C^l}}^{\delta_{kl}} \phi^k_{(x_1)}\, dx_1$$

$$= \underbrace{\left[\int_0^L EI\, \frac{d^2\phi^k}{dx_1^2} \frac{d^2\phi^l}{dx_1^2}\, dx_1\right]}_{\substack{K^{kl} \\ n\times n \text{ matrix}}} \underbrace{C^k}_{\substack{n\times 1 \\ \text{vector}}} + \underbrace{\int_0^L q(x_1)\, \phi^l_{(x_1)}\, dx_1}_{\substack{R^l \\ n\times 1 \text{ vector}}} = 0$$

Choice of ϕ^k:

$\phi^k = x_1^k (L - x_1)$, note they satisfy the essential BCs.

$$\frac{d\phi^k}{dx_1} = k\, x_1^{k-1}(L - x_1) - x_1^k = k\, x_1^{k-1} L - k\, x_1^k - x_1^k$$
$$= k\, x_1^{k-1} L - (k+1)\, x_1^k$$

$$\frac{d^2\phi^k}{dx_1^2} = k(k-1) L\, x^{k-2} - (k+1)k\, x^{k-1}$$

Compute $\displaystyle\int_0^L EI \frac{d^2\phi^k}{dx_1^2}\frac{d^2\phi^\ell}{dx_1^2}\, dx_1 = K^{k\ell}$

$\displaystyle\int_0^L q(x)\, \phi^\ell(x_1)\, dx_1 = R^\ell$ (uniform load

n = 2 $\quad \phi^1 = x_1(L - x_1) \quad \phi^2 = x^2(L - x)$

$$\frac{d^2\phi^1}{dx_1^2} = -2 \qquad \frac{d^2\phi^2}{dx_1^2} = 2L - 6x$$

$$K^{11} = \int_0^L EI \left(\frac{d^2\phi^1}{dx_1^2}\right)^2 dx_1 = \int_0^L EI\, (-2)^2\, dx_1 = 4EIL$$

$$K^{12} = \int_0^L EI \frac{d^2\phi^1}{dx_1^2}\frac{d^2\phi^2}{dx_1^2}\, dx_1 = \int_0^L EI\, (-2)(2L - 6x)\, dx_1 =$$

$$K_{12} = EIL^2(-4) + EI\,12\frac{L^2}{2} = 2EIL^2$$

$$K_{22} = \int_0^L EI \frac{d^2\phi^2}{dx_1^2} dx_1 = \int_0^L EI\,(2L-6x)^2 dx_1 = 4L^3 EI$$

$$R^1 = \int_0^L q\,\phi^1 dx_1 = \frac{qL^3}{6}$$

$$R^2 = \int_0^L q\,\phi^2 dx_1 = \frac{qL^4}{12}$$

$$\longrightarrow \quad EIL \begin{bmatrix} 4 & 2L \\ 2L & 4L^2 \end{bmatrix} \begin{Bmatrix} C_1 \\ C^2 \end{Bmatrix} = \frac{-qL^3}{12} \begin{Bmatrix} 2 \\ L \end{Bmatrix}$$

$$\longrightarrow C_1, C_2$$

$$U(x_1) = \frac{-qL^3}{24EI} x_1 + \frac{qL^2}{24EI} x_1^2$$