16.21 Techniques of Structural Analysis and Design
Spring 2005
Unit #10 - Principle of minimum potential energy and Castigliano's First Theorem

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Principle of minimum potential energy

The principle of virtual displacements applies regardless of the constitutive law. Restrict attention to elastic materials (possibly nonlinear). Start from the PVD:

$$\boxed{\int_V \sigma_{ij}\bar{\epsilon}_{ij} dV = \int_S t_i \bar{u}_i dS + \int_V f_i \bar{u}_i dV, \ \forall \bar{u}/\bar{u} = 0 \text{ on } S_u} \tag{1}$$

Replacing the expression for the stresses for elastic materials:

$$\sigma_{ij} = \frac{\partial U_0}{\partial \epsilon_{ij}}$$

and assuming that the virtual displacement field is a variation of the equilibrated displacement field $\bar{u} = \delta u$, $\bar{\epsilon}_{ij} = \delta\epsilon_{ij}$.

$$\int_V \underbrace{\frac{\partial U_0}{\partial \epsilon_{ij}}\delta\epsilon_{ij}}\, dV = \int_S t_i \delta u_i dS + \int_V f_i \delta u_i dV$$

The expression over the brace is the variation of the strain energy density δU_0:

$$\delta U_0 = \frac{\partial U_0}{\partial \epsilon_{ij}} \delta \epsilon_{ij}$$

Using the properties of calculus of variations $\delta \int () = \int \delta ()$:

$$\int \delta U_0 dV = \delta \int U_0 dV = \delta U = \delta \Big(\int_S t_i u_i dS + \int_V f_i u_i dV \Big) = \delta(-V)$$

where V is the potential of the external loads. Therefore:

$$\boxed{\delta \Pi = \delta(U + V) = 0}$$

which is known as the *Principle of minimum potential energy* (PMPE).

Let's take the reverse path. Starting from the potential energy:

$$\Pi(u_i) = \int_V \frac{1}{2} C_{ijkl} \epsilon_{kl} \epsilon_{ij} dV - \int_S t_i u_i dS - \int_V f_i u_i dV$$

we would like to apply our tools of calculus of variations to find the extrema of Π:

$$\delta_{u_i} \Pi = 0 = \int_V \frac{1}{2} C_{ijkl} (\delta \epsilon_{kl} \epsilon_{ij} + \epsilon_{kl} \delta \epsilon_{ij}) dV - \int_S t_i \delta u_i dS - \int_V f_i \delta u_i dV$$

and, by symmetry of C_{ijkl}:

$$\int_V C_{ijkl} \epsilon_{kl} \delta \epsilon_{ij} dV = \int_S t_i \delta u_i dS + \int_V f_i \delta u_i dV$$

Note that this is the expression of the Principle of Virtual Displacements applied to a linear elastic material.

In fact the expression of the PMPE we derived by setting the variations of $\Pi = 0$ only says that Π is stationary with respect to variations in the displacement field when the body is in equilibrium.

We can prove that it is indeed a minimum in the case of a linear elastic material: $U_0 = \frac{1}{2} C_{ijkl} \epsilon_{kl}$. We want to show:

$$\Pi(v) \geq \Pi(u), \ \forall v$$
$$\Pi(v) = \Pi(u) \Leftrightarrow v = u$$

Consider $\bar{u} = u + \delta u$:

$$\begin{aligned}\Pi(u+\delta u) = &\int_V \left[\frac{1}{2} C_{ijkl}(\epsilon_{ij} + \delta\epsilon_{ij})(\epsilon_{kl} + \delta\epsilon_{kl})\right] dV \\ &- \int_S t_i(u_i + \delta u_i) dS - \int_V F_i(u_i + \delta u_i) dV \\ =&\Pi(u) + \cancel{2}\int_V \frac{1}{\cancel{2}} C_{ijkl}\epsilon_{ij}\delta\epsilon_{kl} dV + \int_V \frac{1}{2} C_{ijkl}\delta\epsilon_{ij}\delta\epsilon_{kl} dV \\ &- \int_S t_i \delta u_i dS - \int_V f_i \delta u_i dV\end{aligned}$$

The second, fourth and fifth term disappear after invoking the PVD and we are left with:

$$\Pi(u+\delta u) = \Pi(u) + \int_V \frac{1}{2} C_{ijkl}\delta\epsilon_{ij}\delta\epsilon_{kl} dV$$

The integral is always ≥ 0, since C_{ijkl} is positive definite. Therefore:

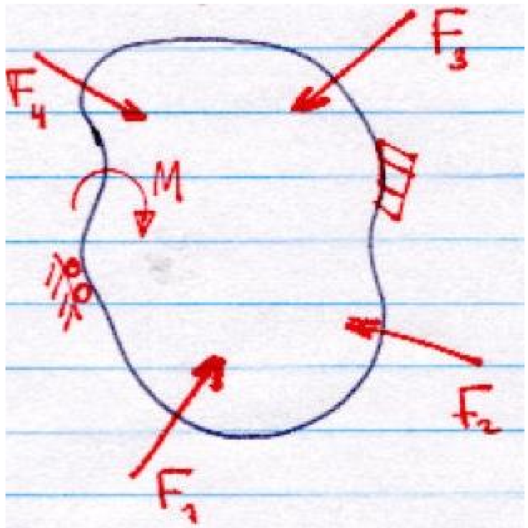
$$\Pi(u+\delta u) = \Pi(u) + a,\ a \geq 0,\ a = 0 \ \Leftrightarrow \delta u = 0$$

and

$$\begin{gathered}\Pi(v) \geq \Pi(u),\ \forall v \\ \Pi(v) = \Pi(u) \Leftrightarrow v = u\end{gathered}$$

as sought.

Castigliano's First theorem



Given a body in equilibrium under the action of N concentrated forces F_I. The potential energy of the external forces is given by:

$$V = -\sum_{I=1}^{N} F_I u_I$$

where the u_I are the values of the displacement field at the point of application of the forces F_I. Imagine that somehow we can express the strain energy as a function of the u_I, i.e.:

$$U = U(u_1, u_2, \ldots, u_N) = U(u_I)$$

Then:

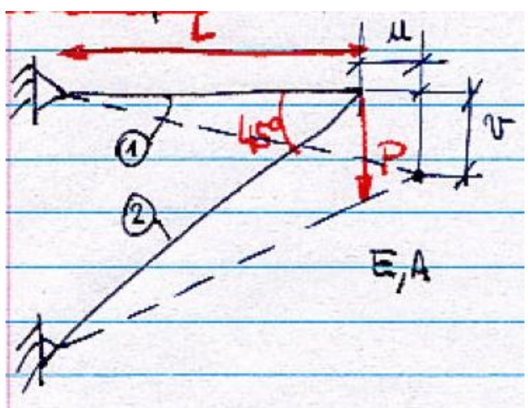
$$\Pi = \Pi(u_I) = U(u_I) + V = U(u_I) - \sum_{I=1}^{N} F_I u_I$$

Invoking the PMPE:

$$
\begin{aligned}
\delta\Pi = 0 = \frac{\partial U}{\partial u_I}\delta u_I - \sum_{I=1}^{N} F_I \underbrace{\frac{\partial u_I}{\partial u_J}} \delta u_J \\
= \frac{\partial U}{\partial u_I}\delta u_I - \sum_{I=1}^{N} F_I \delta_{IJ} \delta u_J \\
= \frac{\partial U}{\partial u_I}\delta u_I - \sum_{I=1}^{N} F_I \delta u_I \\
= \Big(\frac{\partial U}{\partial u_I} - F_I\Big)\delta u_I \\
\forall\ \delta u_I \ \Leftrightarrow \boxed{F_I = \frac{\partial U}{\partial u_I}}
\end{aligned}
$$

Theorem: *If the strain energy can be expressed in terms of N displacements corresponding to N applied forces, the first derivative of the strain energy with respect to displacement u_I is the applied force.*

Example:



$$
\epsilon_I = \sqrt{\frac{(L+u)^2 + v^2}{L^2}} - 1 \sim \frac{u}{L}
$$

$$
\epsilon_I I = \sqrt{\frac{(L+u)^2 + (L-v)^2}{2L^2}} - 1 \sim \frac{1}{2}\frac{u-v}{L}
$$

$$U = \frac{1}{2}\left\{AEL\left(\frac{u}{L}\right)^2 + AE\sqrt{2}L\left[\frac{1}{2}\left(\frac{u-v}{L}\right)\right]^2\right\}$$

Note that we have written $U = U(u, v)$. According to the theorem:

$$0 = \frac{\partial U}{\partial u}$$
$$F = \frac{\partial U}{\partial v}$$

See solution in accompanying mathematica file.
