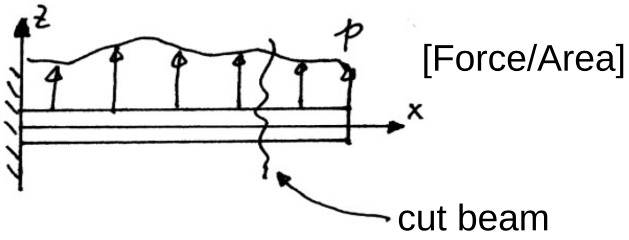
16.20 HANDOUT #4
Fall, 2002
General (Shell) Beam Theory

REVIEW OF SIMPLE BEAM THEORY



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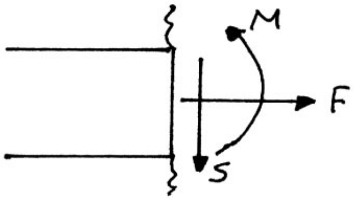


Axial Force: $$F = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} b \, dz$$

Shear Force: $$S = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} b \, dz$$

Bending Moment: $$M = -\int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_{xx} b \, dz$$

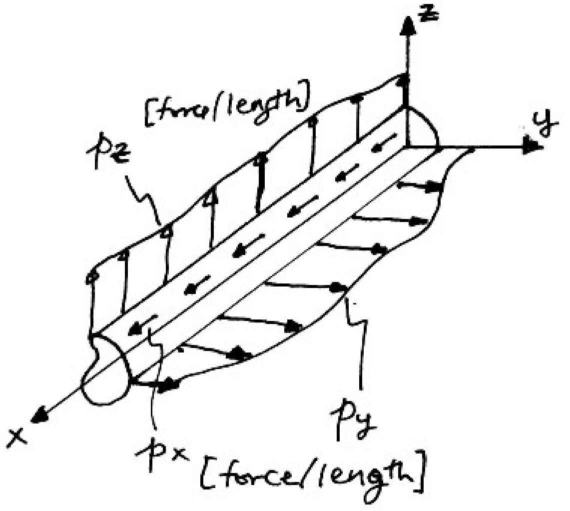
$$\varepsilon_{xx} = -z \frac{d^2 w}{dx^2} = \frac{\partial u}{\partial x}$$

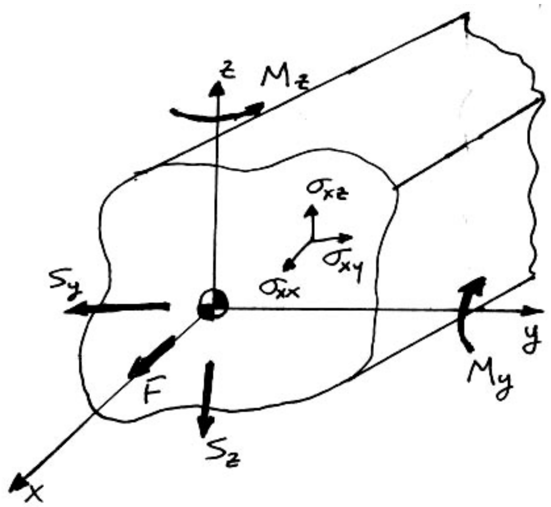
$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \qquad M = EI \frac{d^2 w}{dx^2}$$

$$\frac{dS}{dx} = p \qquad \sigma_{xx} = -\frac{Mz}{I} \qquad I = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 b \, dz$$

$$\frac{dM}{dx} = S \qquad \sigma_{xz} = -\frac{SQ}{Ib} \qquad Q = \int_{z}^{\frac{h}{2}} z b \, dz$$

BEHAVIOR OF GENERAL (INCLUDING UNSYMMETRIC CROSS-SECTION) BEAMS





- Resultants:

$$F = \iint \sigma_{xx} \, dA$$

$$S_y = -\iint \sigma_{xy} \, dA$$

$$S_z = -\iint \sigma_{xz} \, dA$$

$$M_y = -\iint \sigma_{xx} \, z \, dA$$

$$M_z = -\iint \sigma_{xx} \, y \, dA$$

- Displacement:

$$u\,(x,y,z) = u_0 - \underbrace{y \frac{dv}{dx}}_{\substack{\text{bending}\\\text{about}\\\text{z-axis}}} - \underbrace{z \frac{dw}{dx}}_{\substack{\text{bending}\\\text{about}\\\text{y-axis}}}$$

- Strain:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{du_0}{dx} + y\left(-\frac{d^2 v}{dx^2}\right) + z\left(-\frac{d^2 w}{dx^2}\right)$$

$$= f_1 + f_2 y + f_3 z$$

- Strain:

$$\sigma_{xx} = E\left(f_1 + f_2 y + f_3 z\right) - E \alpha \, \Delta T$$

MODULUS - WEIGHTED SECTION PROPERTIES

$$dA^* = \frac{\overset{\substack{\text{modulus of material}\\ \downarrow}}{E}}{\underset{\substack{\uparrow\\ \text{reference modulus}}}{E_1}}\, dA$$

$$\iint y\, dA^* = \bar{y}^* A^*$$

$$\iint z\, dA^* = \bar{z}^* A^*$$

$$\iint dA^* = A^*$$

$$\iint y^2\, dA^* = I_z^*$$

$$\iint z^2\, dA^* = I_y^*$$

$$\iint yz\, dA^* = I_{yz}^*$$

also:

$$\int_{z^{z}}^{\overset{\substack{\text{top location of beam}\\ \swarrow}}{z_T^{*}}} \frac{E}{E_1}\, bzdz = Q^*$$

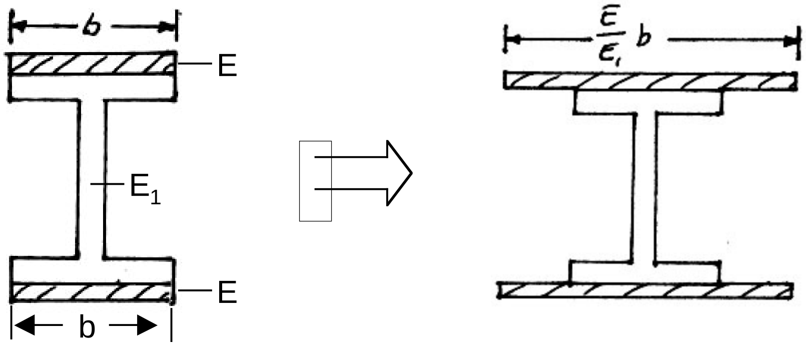
- Thermal Resultants:

$$F^T = \iint E\alpha\Delta T\, dA$$

$$M_y^T = -\iint E\alpha\Delta Tz\, dA$$

$$M_z^T = -\iint E\alpha\Delta Ty\, dA$$

- Selective Reinforcement:



- General Equations:

Mechanical + Thermal Resultants

$$f_1 = \frac{F^{TOT}}{E_1 A^*} = \frac{d u_0}{dx} \qquad \left(F + F^T\right) = F^{TOT}$$

$$-\left(M_z + M_z^T\right) = -M_z^{TOT}$$

$$-\left(M_y + M_y^T\right) = -M_y^{TOT}$$

$$f_2 = \frac{-I_y^* M_z^{TOT} + I_{yz}^* M_y^{TOT}}{E_1\left(I_y^* I_z^* - I_{yz}^{*\,2}\right)} = -\frac{d^2 v}{dx^2}$$

$$f_3 = \frac{-I_z^* M_y^{TOT} + I_{yz}^* M_z^{TOT}}{E_1\left(I_y^* I_z^* - I_{yz}^{*\,2}\right)} = -\frac{d^2 w}{dx^2}$$

$$\sigma_{xx} = \frac{E}{E_1}\left\{\frac{F^{TOT}}{A^*} - \frac{\left[I_y^* M_z^{TOT} - I_{yz}^* M_y^{TOT}\right] y}{I_y^* I_z^* - I_{yz}^{*\,2}}\right.$$

$$\left. - \frac{\left[I_z^* M_y^{TOT} - I_{yz}^* M_z^{TOT}\right] z}{I_y^* I_z^* - I_{yz}^{*\,2}} - E_1 \alpha \Delta T\right\}$$

COMMON SECTIONS

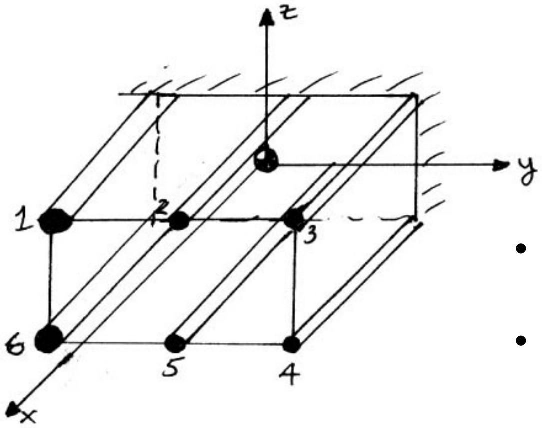
(See Handout #4A)

SHEARING AND TORSION (AND BENDING) OF SHELL BEAMS

Loadings

- axial load: F
- bending moments: M_y, M_z
- shear forces: S_y, S_z
- torque (torsional moment): T

Single Cell "Box Beam" (symmetric)



- use modulus-weighted centroid of flanges
- skin carries only shear loads (no bending/axial loads)

- Location of modulus-weighted centroid

$$\bar{y}^* = \frac{\sum A_i^* \bar{y}_i}{\sum A_i^*} ; \qquad \bar{z}^* = \frac{\sum A_i^* \bar{z}_i}{\sum A_i^*}$$

- Section moments of Inertia

$$I_y^* = \sum A_i^* z_i^{*2}; \quad I_z^* = \sum A_i^* y_i^{*2}; \quad I_{yz}^* = \sum A_i^* y_i^* z_i^*$$

sum over number of flanges (i = 1,…n)

- Stresses in flanges

$$\sigma_{xx} = \frac{E}{E_1} \left\{ \frac{F^{TOT}}{A^*} - E_1 f_2 y - E_1 f_3 z - E_1 \alpha \Delta T \right\}$$

- Shear flow:

$$\underset{\text{shear flow}}{q} = \underset{\text{shear stress}}{\sigma_{xs}}\,\underset{\text{thickness}}{t}$$

(assume shear stress is uniform through thickness)

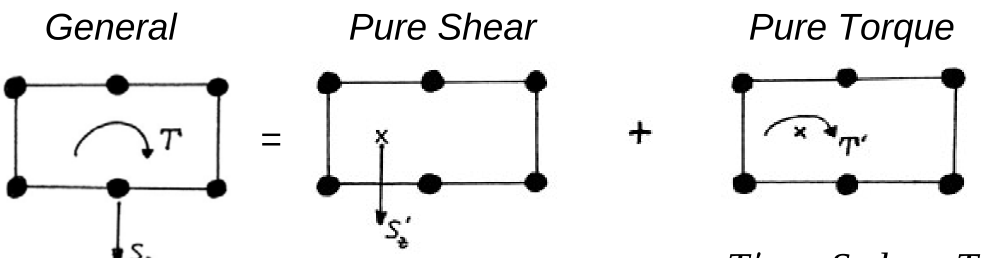
— At any flange (node): $q_{out} - q_{in} = -\frac{dP}{dx}$

for symmetric section ($I_{yz} = 0$) with $M_z = 0$:

$$q_{out} - q_{in} = \frac{Q_y S_z}{I_y}$$

where: $Q_y = A_i z_i^*$ *(moment of area about y)*

- Divide into 2 problems:



$$T' - S_z d = T$$

1. "Pure Shear" Problem

   d = distance from S_z original line of action to shear center

   use:

   → Joint equilibrium: $q_{out} - q_{in} = \frac{Q_y S_z}{I_y}$

   → Torque equivalence: $\sum T_{\text{internal}} = T_{\text{applied}}$

   → No-Twist condition: $\oint \frac{q}{t}\, ds = 0$

2. "Pure Torsion" Problem

   use: → Joint Equilibrium: $q_{out} - q_{in} = 0$

   → Torque equivalence: $\sum T_{\text{internal}} = T_{\text{applied}}$

3. Sum q's from two problems

UNSYMMETRICAL SHELL BEAMS (UNHEATED)

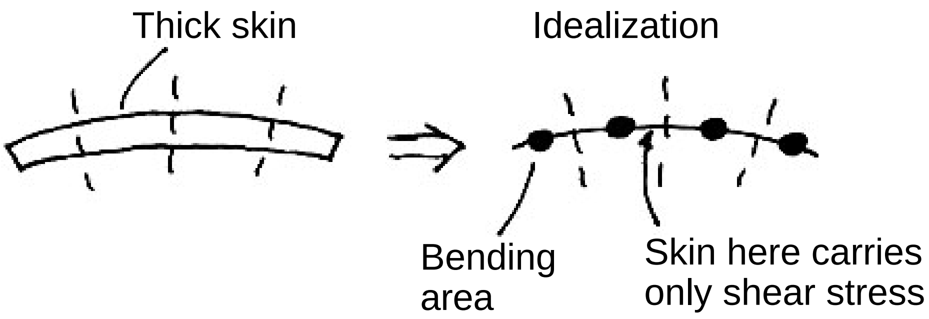
- $$\sigma_{xx} = \frac{\left(-I_y M_z + I_{yz} M_y\right) y + \left(-I_z M_y + I_{yz} M_z\right) z}{I_y I_z - I_{yz}^2}$$

- $$q_{out} - q_{in} = \frac{\left(I_y S_y - I_{yz} S_z\right) Q_z + \left(I_z S_z - I_{yz} S_y\right) Q_y}{I_y I_z - I_{yz}^2}$$

- Same procedure as previous case

THICK SKIN SHELLS

- Idealize by breaking up thick skin into a finite number of bending areas



- Use previous techniques

DEFLECTIONS

deflection of shear center $\begin{cases} v = v_B + v_s \\ w = w_B + w_s \end{cases}$

due to bending *due to shearing*

α = twist/rotation (due to torsion) about shear center

$$\frac{d^2 v_B}{dx^2} = \frac{I_y\, M_z}{E\left(I_y\, I_z - I_{yz}^2\right)} - \frac{I_{yz}\, M_y}{E\left(I_y\, I_z - I_{yz}^2\right)}$$

$$\frac{d^2 w_B}{dx^2} = \frac{I_z M_y}{E\left(I_y I_z - I_{yz}^2\right)} - \frac{I_{yz} M_z}{E\left(I_y I_z - I_{yz}^2\right)}$$

$$\frac{d v_s}{d x} = -\frac{S_y}{G A_{yy}} - \frac{S_z}{G A_{yz}}$$

$$\frac{d w_s}{d x} = -\frac{S_z}{G A_{zz}} - \frac{S_y}{G A_{yz}}$$

$$\frac{d\alpha}{dz} = \frac{T}{GJ}$$

SECTION PROPERTIES

- I_y, I_z, I_{yz} as previously defined:

$$I_z = \sum A^* y^2$$

$$I_y = \sum A^* z^2$$

$$I_{yz} = \sum A^* y z$$

- $$A_{yy} = \frac{1}{\int \frac{\left(\bar{q}_y\right)^2}{t} ds}$$ where: $\left(\bar{q}_y\right.$ = shear flow due to unit S_y

- $$A_{zz} = \frac{1}{\int \frac{\left(\bar{q}_z\right)^2}{t} ds}$$ where: $\left(\bar{q}_z\right.$ = shear flow due to unit S_z

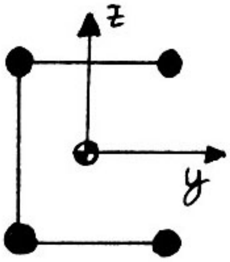
- $$A_{yz} = \frac{1}{\int \frac{\bar{q}_y \bar{q}_z}{t} ds}$$

- $$J = \frac{2A}{\oint \frac{\bar{\bar{q}}}{t} ds}$$ where: $\bar{\bar{q}}$ = shear flow due to unit T

MULTI-CELL SHELL BEAMS

- σ_{xx} equation the same
- q_{out} - q_{in} problem the same
- Additional “No-Twist” Condition -- must be there for each cell as well as overall beam (for “Pure Shear” case)
- “Equal Twist” Condition (for “Pure Torsion” case) -- each cell must twist the same amount. $\left(\frac{d\alpha}{dx} \text{ the same}\right)$

OPEN SECTION SHELL BEAMS



- All q_i determined from joint equilibrium

$$q_{out} - q_{in} = -\frac{dP}{dx}$$

- σ_{xx} equation the same
- Use Membrane Analogy for “Pure Torsion” problem

$$\tau = \frac{2T}{J}x$$