Unit 7
Transformations and Other Coordinate Systems

<u>Readings</u>:

R	2-4, 2-5, 2-7, 2-9
BMP	5.6, 5.7, 5.14, 6.4, 6.8, 6.9, 6.11
T & G	13, Ch. 7 (74 - 83)

On "other" coordinate systems:

T & G	27, 54, 55, 60, 61

Paul A. Lagace, Ph.D.
Professor of Aeronautics & Astronautics
and Engineering Systems

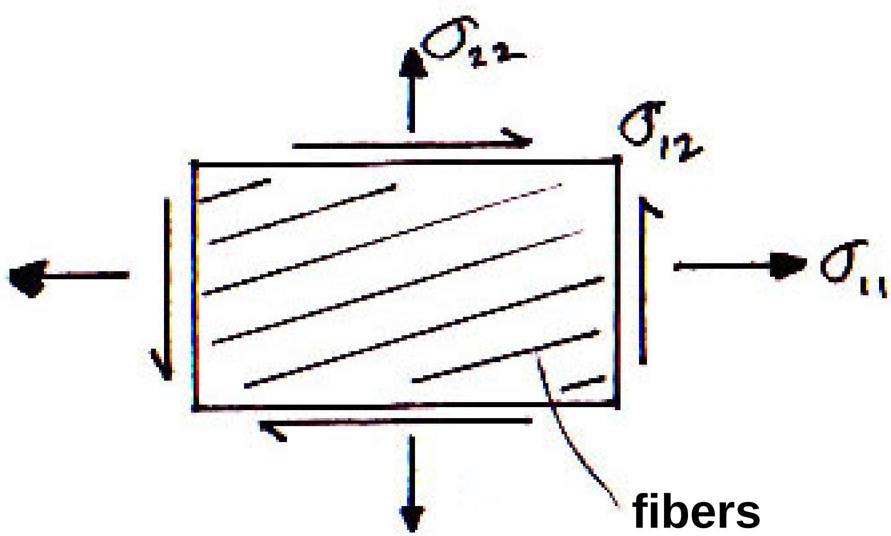
As we've previously noted, we may often want to describe a structure in various axis systems. This involves…

Transformations

(Axis, Deflection, Stress, Strain, Elasticity Tensors)

e.g., loading axes <--> material principal axis

Figure 7.1 **Unidirectional Composite with Fibers at an Angle**



Know stresses along loading axes, but want to know stresses (or whatever) in axis system referenced to the fiber.

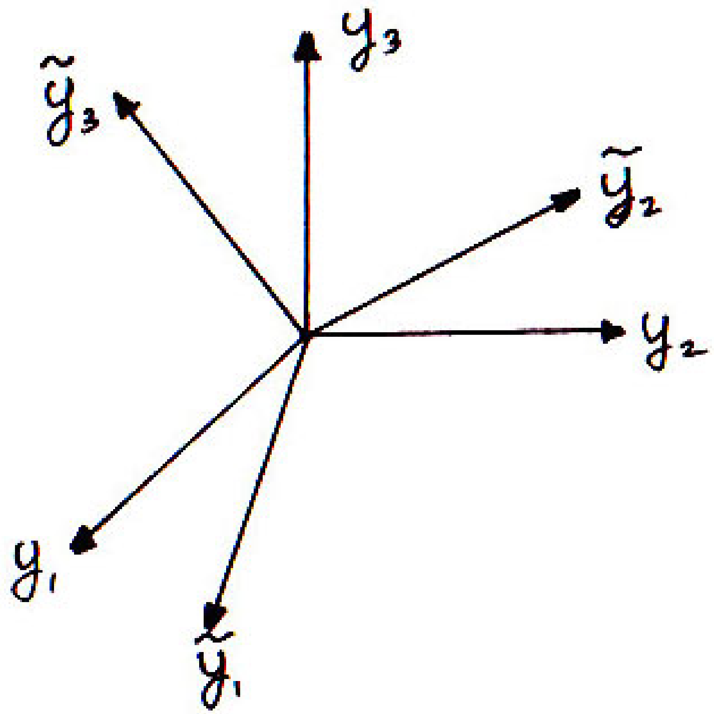
Problem: get expressions for *(whatever)* in one axis system in terms of *(whatever)* in another axis system

(Review from Unified)

Recall: nothing is *inherently* changing, we just describe a body from a different reference.

Use ~ (tilde) to indicate transformed axis system.

Figure 7.2 **General rotation of 3-D rectangular axis system**



ar cartesian)

"Define" this transformation via d<u>irection</u> <u>cosines</u>

$\ell_{\tilde{m}n}$ = cosine of angle from $\tilde{y}_m$ to y_n

<u>Notes</u>: by convention, angle is measured **<u>positive</u>** **<u>counterclockwise</u> (+ CCW)**
(not needed for cosine)

$\ell_{\tilde{m}n} = \ell_{n\tilde{m}}$ since cos is an even function
$\cos(\theta) = \cos(-\theta)$
(reverse direction)

But $\ell_{\tilde{m}n} \neq \ell_{m\tilde{n}}$
angle differs by 2θ!

The order of a tensor governs the transformation needed. An **nth** order tensor requires an **nth** order transformation (can prove by showing link of order of tensor to axis system via governing equations).

Thus:

Quantity	Transformation Equation	Physical Basis
Stress	$\tilde{\sigma}_{mn} = \ell_{\tilde{m}p}\, \ell_{\tilde{n}q}\, \sigma_{pq}$	equilibrium
Strain	$\tilde{\varepsilon}_{mn} = \ell_{\tilde{m}p}\, \ell_{\tilde{n}q}\, \varepsilon_{pq}$	geometry
Axis	$\tilde{x}_m = \ell_{\tilde{m}p}\, x_p$	geometry
Displacement	$\tilde{u}_m = \ell_{\tilde{m}p}\, u_p$	geometry
Elasticity Tensor	$\tilde{E}_{mnpq} = \ell_{\tilde{m}r}\, \ell_{\tilde{n}s}\, \ell_{\tilde{p}t}\, \ell_{\tilde{q}u}\, E_{rstu}$	Hooke's law

In many cases, we deal with 2-D cases

(replace the latin subscripts by greek subscripts)

e.g., $\tilde{\sigma}_{\alpha\beta} = \ell_{\tilde{\alpha}\theta}\, \ell_{\tilde{\beta}\tau}\, \sigma_{\theta\tau}$

(These are written out for 2-D in the handout).

An important way to illustrate transformation of stress and strain in 2-D is via Mohr's circle (recall from Unified). This was actually used B.C. (before calculators). It is a geometrical representation of the transformation.

(See handout).

(you will get to work with this in a problem set).

Also recall…

(Three) <u>Important Aspects Associated with Stress/Strain Transformations</u>

1. <u>Principal Stresses / Strains (Axes)</u>: there is a set of axes into which any state of stress / strain can be resolved such that there are no shear stresses / strains

   --> σ_{ij} depend on applied loads
   --> ε_{ij} depend on applied loads <u>and</u> material response

Thus, <u>note</u>:

For general materials…

axes for principal strain ≠ axes for principal stress

Generally:

(have nothing to do with) material principal axes ≠ principal axes of stress / strain

Find via roots of equation:

$$\begin{vmatrix} \sigma_{11} - \tau & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \tau & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \tau \end{vmatrix} = 0$$

eigenvalues: σ_{I}, σ_{II}, σ_{III}
(same for strain)

2. Invariants: certain combinations of stresses / strains are invariant with respect to the axis system.

Most important: Σ (extensional stresses / strains) = Invariant
very useful in back-of-envelope / "quick check" calculations

3. Extreme shear stresses / strains: (in 3-D) there are three planes along which the shear stresses / strains are maximized. These values are often used in failure analysis (recall Tresca condition from Unified).

These planes are oriented at 45° to the planes defined by the principal axes of stress / strain (use rotation to find these)

Not only do we sometimes want to change the orientation of the axes we use to describe a body, but we find it more convenient to describe a body in a coordinate system other than rectangular cartesian. Thus, consider…

Other Coordinate Systems

The "easiest" case is

Cylindrical (or Polar in 2-D) coordinates

Figure 7.3 **Loaded disk**

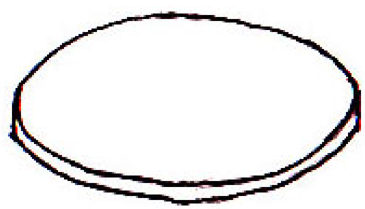


Figure 7.4 **Stress around a hole**

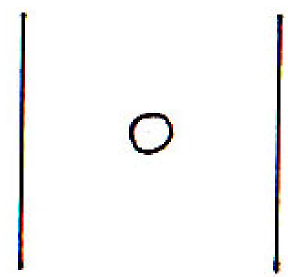
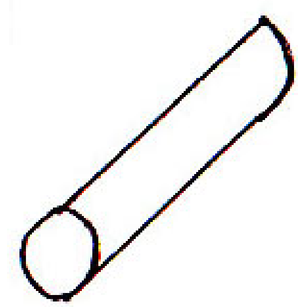


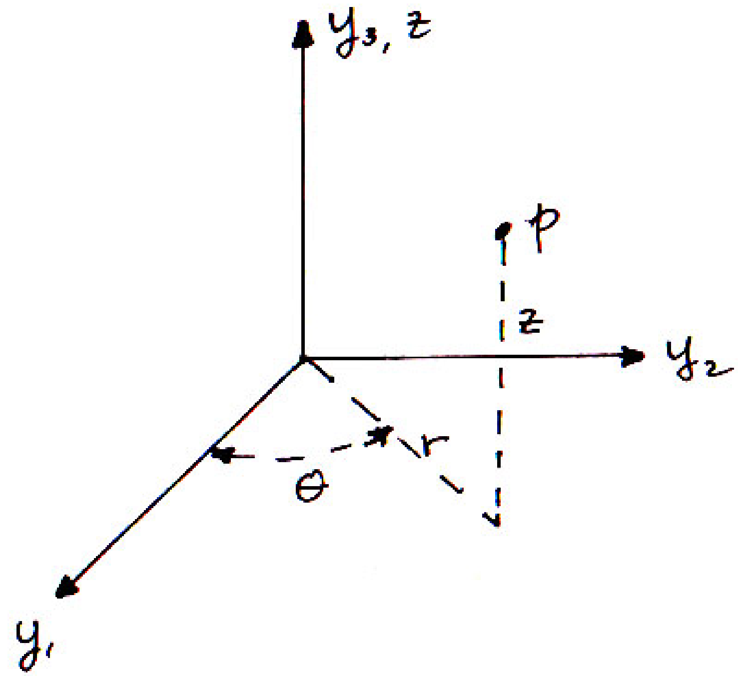
Figure 7.5 **Shaft**



Define the point p by a different set of coordinates other than y_1, y_2, y_3

Figure 7.6 **Polar coordinate representation**

Volume = rd(



Use θ, r, z where:

$y_1 = r \cos\theta$

$y_2 = r \sin\theta$

$y_3 = z$

are the “mapping” functions

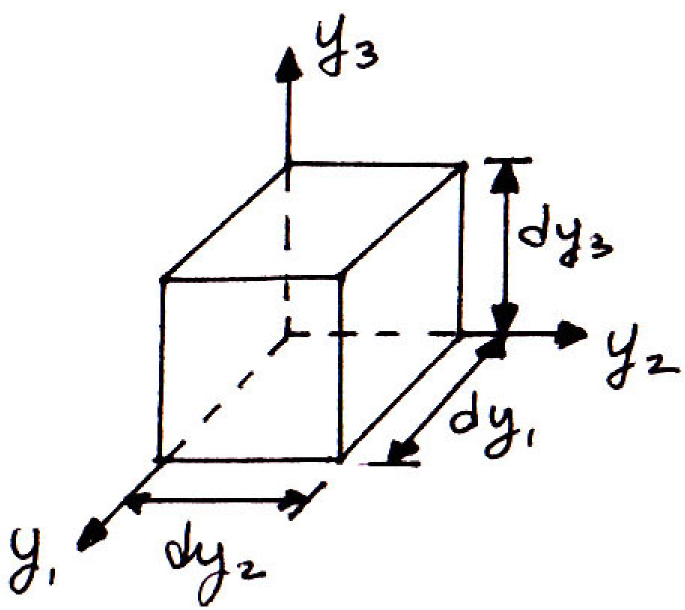
Now the way we describe stresses, etc. change…

--> *<u>Differential element</u>* is now different

<u>Rectangular cartesian</u>

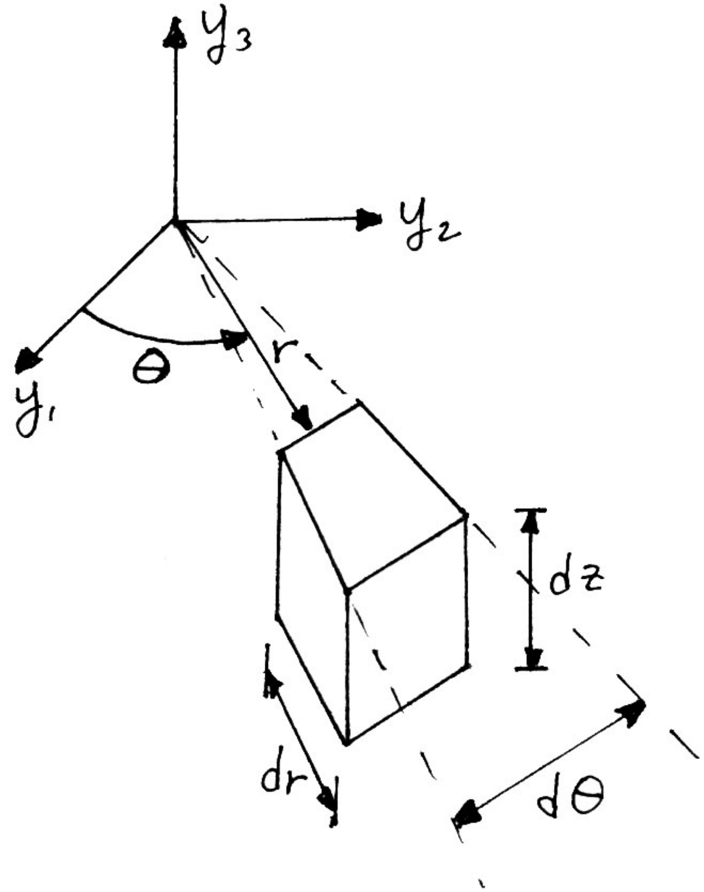
Figure 7.7 **Differential element in rectangular cartesian system**

Volume = dy_1 dy



Cylindrical

Figure 7.8 **Differential element in cylindrical system**



Volume = $r d\theta dr dz$

Generally:

dy_1 --> dr
dy_2 --> $r d\theta$
dy_3 --> dz (get from "mapping" functions)

--> stresses follow same rules:

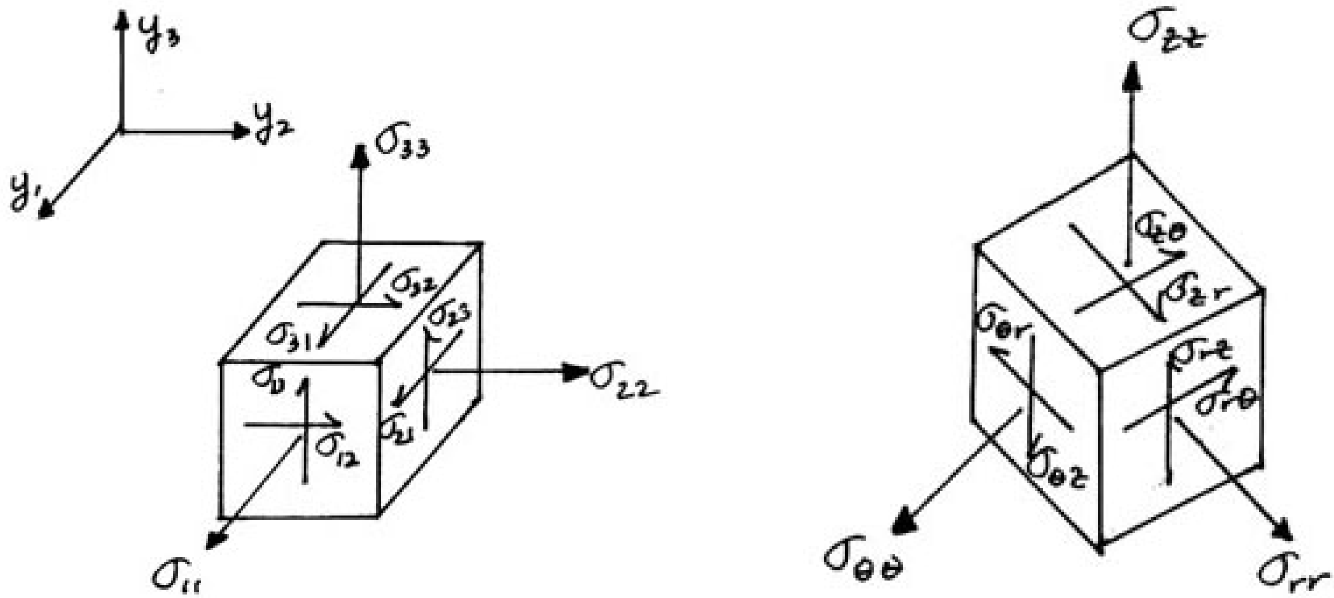
σ_{mn}

m → face

n → direction

but now we deal with r, θ, z faces and directions

Figure 7.9 **Representation of stresses on rectangular cartesian and cylindrical differential elements**



equilibrium considerations (or your "mapping" functions on equations in rectangular cartesian coordinates) yield:

Equilibrium Equations (in cylindrical coordinates)

$$r: \quad \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = 0$$

$$\theta: \quad \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + f_\theta = 0$$

$$z: \quad \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = 0$$

Body forces = f_r, f_θ, f_z

can do a similar manipulation for the

Strain - Displacement Equations

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\varepsilon_{zz} = \frac{\partial u_3}{\partial z}$$

(engineering shear strains)

$$\varepsilon_{r\theta} = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}$$

$$\varepsilon_{\theta z} = \frac{1}{r}\frac{\partial u_3}{\partial \theta} + \frac{\partial u_\theta}{\partial z}$$

$$\varepsilon_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_3}{\partial r}$$

Stress - Strain Equations become, however, more complicated and not easy to "map" into another coordinate systems.

Why? Unless the material is isotropic, the properties change with direction (if the material principal axes are rectangular orthogonal).

$$\tilde{E}_{mnpq} = \ell_{\tilde{m}r}\,\ell_{\tilde{n}s}\,\ell_{\tilde{p}t}\,\ell_{\tilde{q}u}\,E_{rstu}$$

So, for cylindrical coordinates:

$E_{mnpq}\,(\theta)$

↑ a function of θ

Think of it this way:

In cylindrical (polar 2-D) coordinates, we have rotated a "local rectangular cartesian system". So we use the $\tilde{E}_{mnpq}$ transformation with the angle θ to find the elasticity tensor values and then the "local" engineering constants.

Recall / note: a material that is orthotropic may pick up additional coupling terms in this rotation and "appear" anisotropic in that local coordinate system.

In the isotropic case, can write:

$$\varepsilon_{rr} = \frac{1}{E}\left[\sigma_{rr} - \nu\left(\sigma_{\theta\theta} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\left(\sigma_{rr} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right]$$

$$\varepsilon_{r\theta} = \frac{2(1 + \nu)}{E} \sigma_{r\theta}$$

$$\varepsilon_{\theta z} = \frac{2(1 + \nu)}{E} \sigma_{\theta z}$$

$$\varepsilon_{zr} = \frac{2(1 + \nu)}{E} \sigma_{zr}$$

More generally, we can express transformation to any…

General Curvilinear Coordinates

(including locally non-rectangular Cartesian systems)

functional forms:

$$F_1 (y_1, y_2, y_3) = \xi$$

$$F_2 (y_1, y_2, y_3) = \eta$$

$$F_3 (y_1, y_2, y_3) = \zeta$$

Use these to “transform” governing equations from basic rectangular case

For cylindrical case:

$\xi = r$

$\eta = \theta$

$\zeta = z$

actual "mapping" functions:

$$F_1 (y_1, y_2, y_3) = \sqrt{y_1^2 + y_2^2}$$

$$F_2 (y_1, y_2, y_3) = \tan^{-1} \left(y_2 / y_1 \right)$$

$$F_3 (y_1, y_2, y_3) = y_3$$

Other cases

Let's next consider some general solution approaches