Unit 3

(Review of) Language of Stress/Strain Analysis

Readings:

B, M, P	A.2, A.3, A.6
Rivello	2.1, 2.2
T & G	Ch. 1 (especially 1.7)

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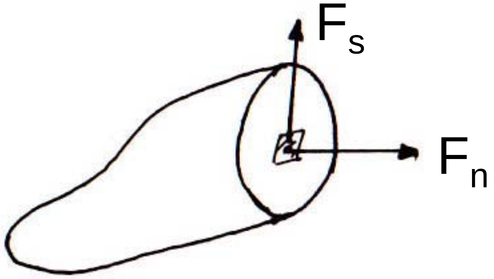
Recall the definition of stress:

σ = stress = “intensity of internal force at a point”

Figure 3.1 **Representation of cross-section of a general body**



$$\text{Stress} = \lim_{\Delta A \to 0} \left(\frac{\Delta F}{\Delta A}\right)$$

There are two types of stress:

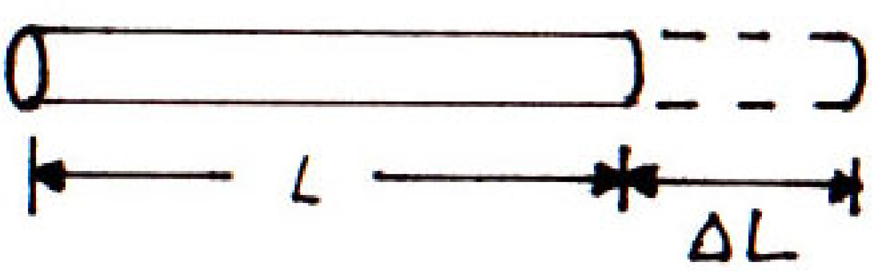
- σ_n (F_n) 1. Normal (or extensional): act normal to the plane of the element
- σ_s (F_s) 2. Shear: act in-plane of element

$\rightarrow$ Sometimes delineated as τ

And recall the definition of strain:

ε = strain = “percentage deformation of an infinitesimal element”

Figure 3.2 **Representation of 1-Dimensional Extension of a body**



$$\varepsilon = \lim_{L \to 0} \left(\frac{\Delta L}{L} \right)$$

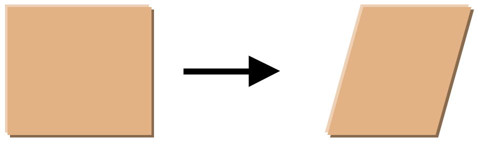
Again, there are two types of strain:

ε_n 1. Normal (or extensional): elongation of element

ε_s 2. Shear: angular change of element

→ Sometimes delineated as γ

Figure 3.3 **Illustration of Shear Deformation**



shear deformation!

Since stress and strain have components in several directions, we need a notation to represent these (as you learnt initially in Unified)

Several possible

- Tensor (indicial) notation
- Contracted notation
- Engineering notation
- Matrix notation

$\left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\}$ *will review here and give examples in recitation*

IMPORTANT: Regardless of the notation, the equations and concepts have the same meaning

⇒ *learn, be comfortable with, be able to use all notations*

Tensor (or Summation) Notation

- "Easy" to write complicated formulae
- "Easy" to mathematically manipulate
- "Elegant", rigorous
- Use for derivations or to succinctly express a set of equations or a long equation

Example: $x_i = f_{ij} y_j$

- Rules for subscripts

NOTE: index ≡ subscript

  - Latin subscripts (m, n, p, q, …) take on the values 1, 2, 3 (3-D)
  - Greek subscripts (α, β, γ …) take on the values 1, 2 (2-D)
  - When subscripts are repeated on one side of the equation within one term, they are called *dummy indices* and are to be summed on

Thus:

$$f_{ij} y_j = \sum_{j=1}^{3} f_{ij} y_j$$

But $f_{ij} y_j + g_i$ ... **do not sum on i !**

  - Subscripts which appear only once on the left side of the equation within one term are called free indices and represent a separate equation

Thus:

x_i = …..

$\Rightarrow$ x_1 = …..

x_2 = …..

x_3 = …..

Key Concept: The letters used for indices have no inherent meaning in and of themselves

Thus: $x_i = f_{ij}\, y_j$

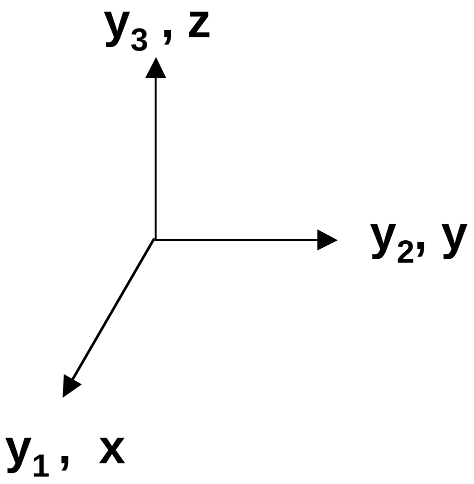
is the same as: $x_r = f_{rs}\, y_s$ **or** $x_j = f_{ji}\, y_i$

Now apply these concepts for stress/strain analysis:

1. Coordinate System

Generally deal with right-handed rectangular Cartesian: y_m

Figure 3.4 Right-handed rectangular Cartesian coordinate system



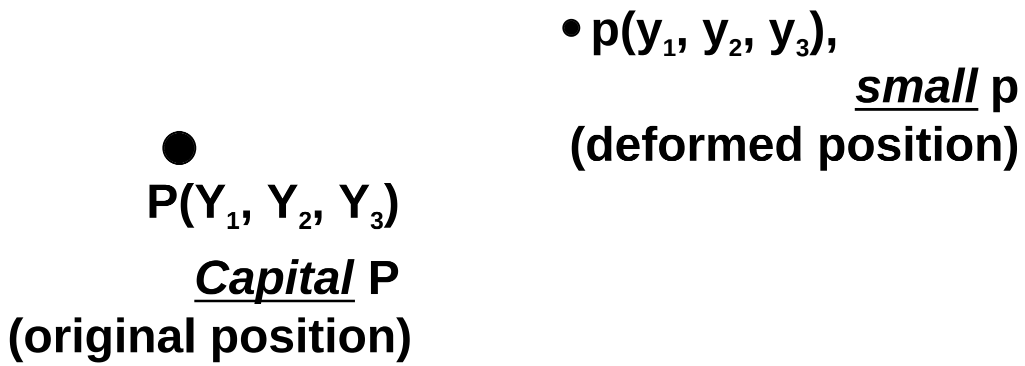
Compare notations

Tensor	Engineering
y_1	x
y_2	y
y_3	z

Note: *Normally* this is so, but always check definitions in any article, book, report, etc. Key issue is self-consistency, not consistency with a worldwide standard (an official one does *not* exist!)

2. Deformations/Displacements (3)

Figure 3.5



• $p(y_1, y_2, y_3)$,
small **p**
(deformed position)

$P(Y_1, Y_2, Y_3)$
Capital **P**
(original position)

$$u_m = p(y_m) - P(y_m)$$

--> Compare notations

Tensor	**Engineering**	**Direction in Engineering**
u_1	u	x
u_2	v	y
u_3	w	z

3. Components of Stress (6)

σ_{mn} "Stress Tensor"

2 subscripts ⇒ 2nd order tensor

6 independent components

Extensional	Shear
σ_{11}	$\sigma_{12} = \sigma_{21}$
σ_{22}	$\sigma_{23} = \sigma_{32}$
σ_{33}	$\sigma_{13} = \sigma_{31}$

Note: stress tensor is symmetric

$$\sigma_{mn} = \sigma_{nm}$$

due to equilibrium (moment) considerations

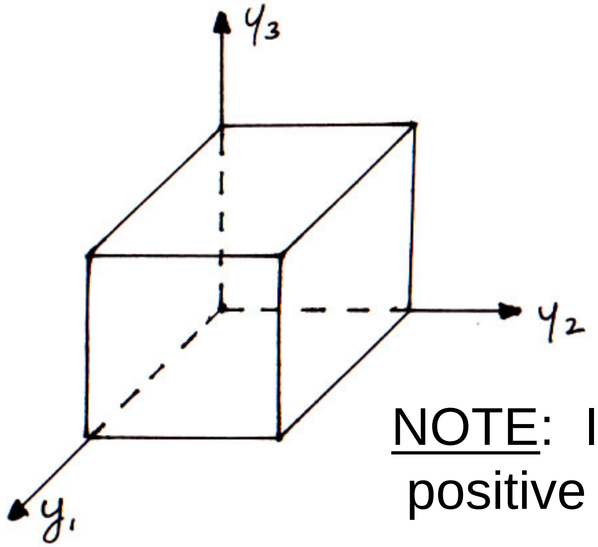
Meaning of subscripts:

$\boldsymbol{\sigma_{mn}}$

stress acts in n-direction

stress acts on face with normal vector in the m-direction

Figure 3.6 Differential element in rectangular system



<u>NOTE</u>: If face has a "negative normal", positive stress is in negative direction

--> <u>Compare notations</u>

Tensor	Engineering	
σ_{11}	σ_x	
σ_{22}	σ_y	
σ_{33}	σ_z	
σ_{23}	σ_{yz}	$= \tau_{yz}$
σ_{13}	σ_{xz}	$= \tau_{xz}$
σ_{12}	σ_{xy}	$= \tau_{xy}$

} sometimes used for shear stresses

4. Components of Strain (6)

ε_{mn} "Strain Tensor"

2 subscripts ⇒ 2nd order tensor

6 independent components

Extensional

ε_{11}

ε_{22}

ε_{33}

Shear

$\varepsilon_{12} = \varepsilon_{21}$

$\varepsilon_{23} = \varepsilon_{32}$

$\varepsilon_{13} = \varepsilon_{31}$

NOTE (again): strain tensor is symmetric

$$\varepsilon_{mn} = \varepsilon_{nm}$$

due to geometrical considerations

(from Unified)

Meaning of subscripts *not* like stress

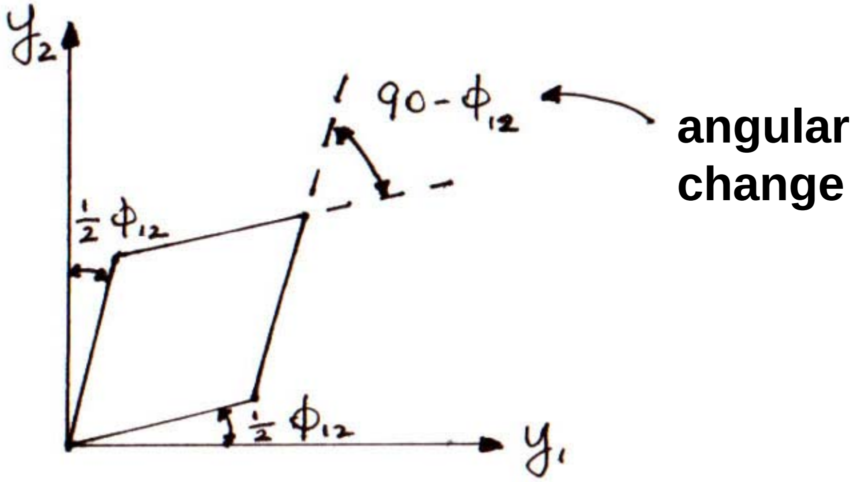
ε_{mn}

m = n ⇒ extension along m
m ≠ n ⇒ rotation in m-n plane

BIG DIFFERENCE for strain tensor:

There is a difference in the shear components of strain between tensor and engineering (unlike for stress).

Figure 3.7 **Representation of shearing of a 2-D element**



--> total angular change = $\phi_{12} = \varepsilon_{12} + \varepsilon_{21} = \underline{2}\ \varepsilon_{12}$

(recall that ε_{12} and ε_{21} are the same due to geometrical considerations)

<u>But</u>, engineering shear strain is the total angle: $\phi_{12} = \varepsilon_{xy} = \gamma_{xy}$

--> <u>Compare</u> <u>notations</u>

Tensor	**Engineering**	
ε_{11}	ε_x	
ε_{22}	ε_y	
ε_{33}	ε_z	
$2\varepsilon_{23}$ =	ε_{yz}	$= \gamma_{yz}$
$2\varepsilon_{13}$ =	ε_{xz}	$= \gamma_{xz}$
$2\varepsilon_{12}$ =	ε_{xy}	$= \gamma_{xy}$

} sometimes used for shear strains

Thus, factor of 2 will pop up

When we consider the equations of elasticity, the 2 comes out naturally.

(But, remember this “physical” explanation)

CAUTION

When dealing with shear strains, must know if they are tensorial or engineering…DO NOT ASSUME!

5. Body Forces (3)

f_i internal forces act along axes

(resolve them in this manner -- can always do that)

--> Compare notations

Tensor	Engineering
f_1	f_x
f_2	f_y
f_3	f_z

6. Elasticity Tensor (? … will go over later)

E_{mnpq} relates stress and strain

(we will go over in detail, … recall introduction in Unified)

Other Notations

Engineering Notation

- One of two most commonly used
- Requires writing out all equations (no “shorthand”)
- Easier to see all components when written out fully

Contracted Notation

- Other of two most commonly used
- Requires less writing
- Often used with composites (“reduces” four subscripts on elasticity term to two)
- Meaning of subscripts not as “physical”
- Requires writing out all equations generally (there is contracted “shorthand”)

--> subscript changes

Tensor	**Engineering**	**Contracted**
11	x	1
22	y	2
33	z	3
23, 32	yz	4
13, 31	xz	5
12, 21	xy	6

--> Meaning of “4, 5, 6” in contracted notation

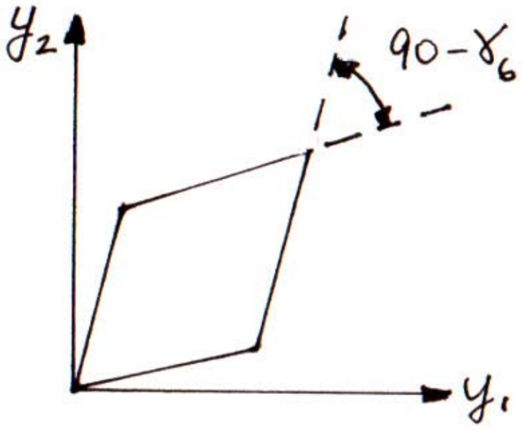
- Shear component
- Represents axis (x_n) “about which” shear rotation takes place via:

$$m = 3 + n$$

γ_m or τ_m → m

x_n → n

Figure 3.8 **Example:**
Rotation about $y_{\underline{3}}$



Matrix notation

- "Super" shorthand
- Easy way to represent system of equations
- Especially adaptable with indicial notation
- Very useful in manipulating equations (derivations, etc.)

Example: $x_i = A_{ij} y_j$

$$\underset{\sim}{x} = \underset{\sim}{A}\,\underset{\sim}{y}$$

$\sim$ ⇒ matrix (as underscore)

tilde

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$$

(will see a little of this … mainly in 16.21)

KEY: Must be able to use various notations. Don't rely on notation, understand concept that is represented.