Unit 23
Vibration of Continuous Systems

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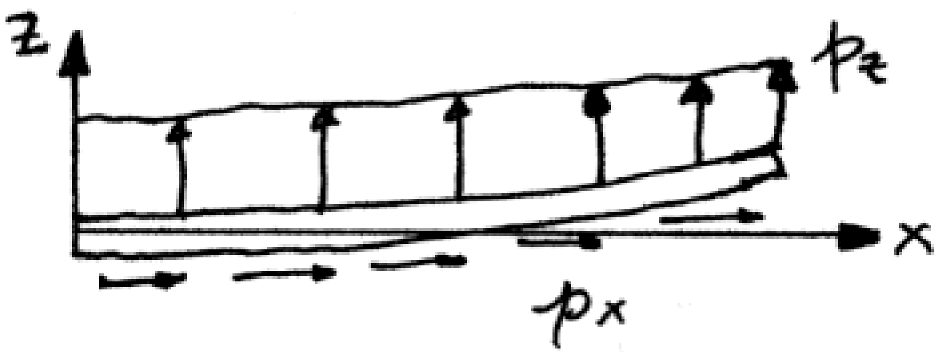
The logical extension of discrete mass systems is one of an infinite number of masses. In the limit, this is a continuous system.

Take the generalized beam-column as a generic representation:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - \frac{d}{dx}\left(F\frac{dw}{dx}\right) = p_z \qquad (23\text{-}1)$$

Figure 23.1 **Representation of generalized beam-column**



$$\frac{dF}{dx} = -p_x$$

This considers only static loads. Must add the inertial load(s). Since the concern is in the z-displacement (w):

$$\text{Inertial load/unit length} = m\ddot{w} \qquad (23\text{-}2)$$

where: m(x) = mass/unit length

Use <u>per</u> <u>unit</u> <u>length</u> since entire equation is of this form. Thus:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - \frac{d}{dx}\left(F\frac{dw}{dx}\right) = p_z - m\ddot{w}$$

or:

$$\boxed{\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - \frac{d}{dx}\left(F\frac{dw}{dx}\right) + m\ddot{w} = p_z} \qquad (23\text{-}3)$$

Beam Bending Equation

often, F = 0 and this becomes:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) + m\ddot{w} = p_z$$

--> This is a **fourth** order differential equation in x

--> Need **four** boundary conditions

--> This is a **second** order differential equation in time

--> Need **two** initial conditions

Notes:

- Could also get via simple beam equations. Change occurs in:

$$\frac{dS}{dx} = p_z - m\ddot{w}$$

- If consider dynamics along x, must include $m\ddot{u}$ in p_x term: ($p_x - m\ddot{u}$)

Use the same approach as in the discrete spring-mass systems:

Free Vibration

Again assume harmonic motion. In a continuous system, there are an infinite number of natural frequencies (eigenvalues) and associated modes (eigenvectors)

so:

$$w(x, t) = \bar{w}(x)\, e^{i\omega t}$$

separable solution spatially (x) and temporally (t)

Consider the homogeneous case ($p_z = 0$) and let there be no axial forces ($p_x = 0 \Rightarrow F = 0$)

So:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 w}{dx^2}\right) + m\ddot{w} = 0$$

Also assume that EI does not vary with x:

$$EI\frac{d^4 w}{dx^4} + m\ddot{w} = 0 \qquad (23\text{-}5)$$

Placing the assumed mode in the governing equation:

$$EI\frac{d^4 \overline{w}}{dx^4} e^{i\omega t} - m\omega^2 \overline{w} e^{i\omega t} = 0$$

This gives:

$$EI\frac{d^4 \overline{w}}{dx^4} - m\omega^2 \overline{w} = 0 \qquad (23\text{-}6)$$

which is now an equation solely in the spatial variable (successful separation of t and x dependencies)

Must now find a solution for $\overline{w}(x)$ which satisfies the differential equations and the boundary conditions.

Note: the shape and frequency are *intimately* linked (through equation 23-6)

Can recast equation (23-6) to be:

$$\frac{d^4 \overline{w}}{dx^4} - \frac{m\omega^2}{EI}\overline{w} = 0 \qquad (23\text{-}7)$$

The solution to this homogeneous equation is of the form:

$$\overline{w}(x) = e^{px}$$

Putting this into (23-7) yields

$$p^4 e^{px} - \frac{m\omega^2}{EI} e^{px} = 0$$

$$\Rightarrow \quad p^4 = \frac{m\omega^2}{EI}$$

So this is an eigenvalue problem (spatially). The four roots are:

$p = +\lambda, -\lambda, +i\lambda, -i\lambda$

where:

$$\lambda = \left(\frac{m\omega^2}{EI}\right)^{1/4}$$

This yields:

$$\bar{w}(x) = Ae^{\lambda x} + Be^{-\lambda x} + Ce^{i\lambda x} + De^{-i\lambda x}$$

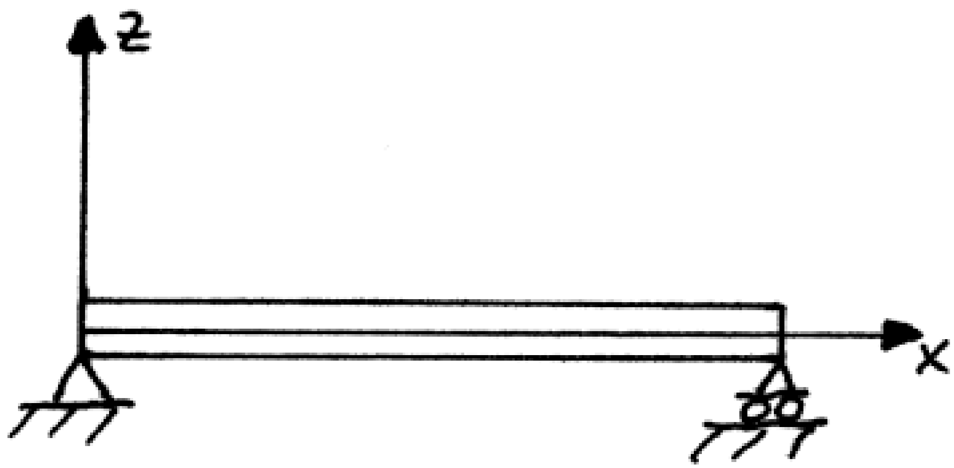
or:

$$\bar{w}(x) = C_1 \sinh \lambda x + C_2 \cosh \lambda x + C_3 \sin \lambda x + C_4 \cos \lambda x \qquad (23\text{-}8)$$

The constants are found by applying the boundary conditions (4 constants ⇒ 4 boundary conditions)

Example: Simply-supported beam

Figure 23.2 **Representation of simply-supported beam**



EI, m = constant with x

$$EI \frac{d^4 w}{dx^4} + m \frac{d^2 w}{dt^2} = p_z$$

Boundary conditions:

@ x = 0 $\left\{ \begin{array}{l} w = 0 \\ M = EI \dfrac{d^2 w}{dx^2} = 0 \end{array} \right.$

@ x = ℓ

with:

$$\bar{w}(x) = C_1 \sinh \lambda x + C_2 \cosh \lambda x + C_3 \sin \lambda x + C_4 \cos \lambda x$$

Put the resulting four equations in matrix form

$$w(0) = 0, \quad \frac{d^2 w}{dx^2}(0) = 0, \quad w(l) = 0, \quad \frac{d^2 w}{dx^2}(l) = 0$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ \sinh \lambda l & \cosh \lambda l & \sin \lambda l & \cos \lambda l \\ \sinh \lambda l & \cosh \lambda l & -\sin \lambda l & -\cos \lambda l \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solution of determinant matrix generally yields values of λ which then yield frequencies and associated modes (as was done for multiple mass systems in a somewhat similar fashion)

In this case, the determinant of the matrix yields:

$$C_3 \sin \lambda l = 0$$

<u>Note</u>: Equations (1 & 2) give $C_2 = C_4 = 0$
Equations (3 & 4) give $2 C_3 \sin \lambda l = 0$
$\Rightarrow$ nontrivial: $\lambda \ell = n\pi$

The nontrivial solution is:

$\lambda \ell = n\pi$ (eigenvalue problem!)

Recalling that:

$$\lambda = \left(\frac{m\omega^2}{EI} \right)^{1/4}$$

$$\Rightarrow \frac{m\omega^2}{EI} = \frac{n^4 \pi^4}{l^4}$$

(change n to r to be consistent with previous notation)

$$\Rightarrow \boxed{\omega_r = r^2 \pi^2 \sqrt{\frac{EI}{m l^4}}}$$

<-- natural frequency

As before, find associated mode (eigenvector), by putting this back in the governing matrix equation.

Here (setting C_3 = 1…..one “arbitrary” magnitude):

$$\bar{w}(x) = \phi_r = \sin \frac{r\pi x}{l}$$

<-- mode shape (normal mode) for: r = 1, 2, 3,……∞

<u>Note</u>: A continuous system has an <u>**infinite**</u> number of modes

So total solution is:

$$w(x, t) = \phi_r \sin \omega_r t = \sin \frac{r\pi x}{l} \sin\left(r^2 \pi^2 \sqrt{\frac{EI}{ml^4}}\, t \right)$$

--> Vibration modes and frequencies are:

Figure 23.3 Representation of vibration modes of simply-supported beam

1st mode

$$\omega_1 = \pi^2 \sqrt{\frac{EI}{ml^4}}$$

2nd mode

$$\omega_2 = 4\pi^2 \sqrt{\frac{EI}{ml^4}}$$

3rd mode

$$\omega_3 = 9\pi^2 \sqrt{\frac{EI}{ml^4}}$$

etc.

Same for other cases

Continue to see the similarity in results between continuous and multi-mass (degree-of-freedom) systems. Multi-mass systems have predetermined modes since discretization constrains system to deform accordingly.

The extension is also valid for…

Orthogonality Relations

They take basically the same form except now have *continuous functions integrated* spatially over the regime of interest *rather than vectors*:

$$\int_0^l m(x)\, \phi_r(x)\, \phi_s(x)\, dx = M_r\, \delta_{rs} \qquad (23\text{-}9)$$

where:

δ_{rs} = kronecker delta $\begin{cases} = 1 & \text{for } r = s \\ = 0 & \text{for } r \neq s \end{cases}$

$M_r = \int_0^l m(x)\, \phi_r^2(x)\, dx$

M_r ← generalized mass of the rth mode

So:

$$\int_0^l m\,\phi_r\,\phi_s\,dx = 0 \qquad r \neq s$$

$$\int_0^l m\,\phi_r\,\phi_r\,dx = M_r$$

Also can show (similar to multi degree-of-freedom case):

$$\int_0^l \frac{d^2}{dx^2}\left(EI\frac{d^2\phi_r}{dx^2}\right)\phi_s\,dx = \delta_{rs}\,M_r\,\omega_r^{\,2} \qquad \text{(23-10)}$$

This again, leads to the ability to transform the equation based on the normal modes to get the…

Normal Equations of Motion

Let:

$$w(x,t) = \sum_{r=1}^{\infty}\phi_r(x)\,\xi_r(t) \qquad \text{(23-11)}$$

($\phi_r(x)$: normal mode; $\xi_r(t)$: normal coordinates)

Place into governing equation:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 w}{dx^2}\right) + m\frac{d^2 w}{dt^2} = p_z(x)$$

multiply by ϕ_s and integrate $\int_0^l \; dx$ to get:

$$\sum_{r=1}^{\infty} \ddot{\xi}_r \int_0^l m\,\phi_r\,\phi_s\,dx + \sum_{r=1}^{\infty} \xi_r \int_0^l \phi_s \frac{d^2}{dx^2}\left(EI\frac{d^2\phi_r}{dx^2}\right) dx = \int_0^l \phi_s f\,dx$$

Using orthogonality conditions, this takes on the same forms as before:

$$\boxed{M_r\,\ddot{\xi}_r + M_r\,\omega_r^{\,2}\xi_r = \Xi_r} \qquad (23\text{-}12)$$

r = 1, 2, 3,……∞

with: $M_r = \int_0^l m\,\phi_r^{\,2}\,dx$ - Generalized mass of rth mode

$\Xi_r = \int_0^l \phi_r\,p_z(x, t)\,dx$ - Generalized force of rth mode

ξ_r (t) = normal coordinates

Once again

- each equation can be solved independently
- allows continuous system to be treated as a series of "simple" one degree-of-freedom systems
- superpose solutions to get total response (Superposition of Normal Modes)
- often only lowest modes are important
- difference from multi degree-of-freedom system: n --> ∞

--> To find Initial Conditions in normalized coordinates…same as before:

$$w(x, 0) = \sum_r \phi_r(x)\, \xi_r(0)$$

etc.

Thus:

$$\xi_r(0) = \frac{1}{M_r} \int_0^l m\, \phi_r\, w_0(x)\, dx$$

$$\dot{\xi}_r(0) = \frac{1}{M_r} \int_0^l m\, \phi_r\, \dot{w}_0(x)\, dx \qquad (23\text{-}13)$$

Finally, can add the case of…

Forced Vibration

Again… response is made up of the natural modes

- Break up force into series of spatial impulses
- Use Duhamel's (convolution) integral to get response for each normalized mode

$$\xi_r(t) = \frac{1}{M_r \omega_r} \int_0^t \Xi_r(\tau) \sin \omega_r (t - \tau)\, d\tau \qquad (23\text{-}14)$$

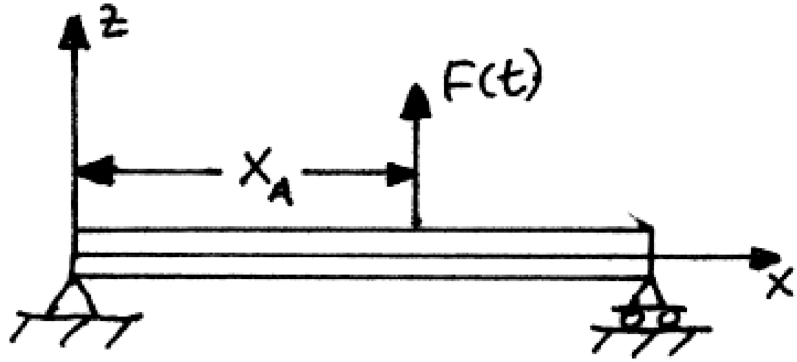
- Add up responses (equation 23-11) for all normalized modes

(Linear ⇒ Superposition)

What about the special case of…

--> Sinusoidal Force at point x_A

Figure 23.4 **Representation of force at point x_A on simply-supported beam**



$$F(t) = F_o \sin \Omega t$$

As for single degree-of-freedom system, for each normal mode get:

$$\xi_r(t) = \frac{\phi_r(x_A) F_o}{M_r \omega_r^2 \left(1 - \frac{\Omega^2}{\omega_r^2}\right)} \sin \Omega t$$

for steady state response (Again, initial transient of sin ω_rt dies out due to damping)

Add up all responses…

Note:

- Resonance can occur at any ω_r
- DMF (Dynamic Magnification Factor) associated with each normal mode

--> Can apply technique to any system.

- Get governing equation *including* inertial terms
- Determine Free Vibration Modes and frequencies
- Transform equation to uncoupled single degree-of-freedom system (normal equations)
- Solve each normal equation separately
- Total response equal to sum of individual responses

Modal superposition is a very powerful technique!