Unit 21
Influence Coefficients

Readings:

Rivello 6.6, 6.13 (again), 10.5

Paul A. Lagace, Ph.D.
Professor of Aeronautics & Astronautics
and Engineering Systems

Have considered the vibrational behavior of a discrete system. How does one use this for a continuous structure?

First need the concept of…..

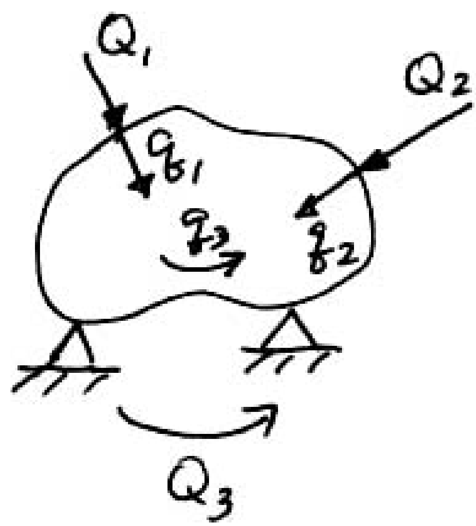
<u>Influence</u> <u>Coefficients</u>

which tell how a force/displacement at a particular point "influences" a displacement/force at another point

--> useful in matrix methods…
- finite element method
- lumped mass model (will use this in next unit)

--> consider an arbitrary elastic body and define:

Figure 21.1 **Representation of general forces on an arbitrary elastic body**



q_i = generalized displacement (linear or rotation)

Q_i = generalized force (force or moment/torque)

Note that Q_i and q_i are:

- at the same point
- have the same sense (i.e. direction)
- of the same "type"

  (force ↔ displacement)

  (moment ↔ rotation)

For a linear, elastic body, superposition applies, so can write:

$$q_i = C_{ij} Q_j \begin{cases} q_1 = C_{11} Q_1 + C_{12} Q_2 + C_{13} Q_3 \\ q_2 = C_{21} Q_1 + C_{22} Q_2 + C_{23} Q_3 \\ q_3 = C_{31} Q_1 + C_{32} Q_2 + C_{33} Q_3 \end{cases}$$

or in Matrix Notation:

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

<u>Note</u>: $\lfloor \quad \rfloor$ --> row
$\{ \quad \}$ --> column
$[\quad]$ --> full matrix

<u>or</u>

$$\left\{ q_i \right\} = \left[C_{ij} \right] \left\{ Q_j \right\}$$

<u>or</u>

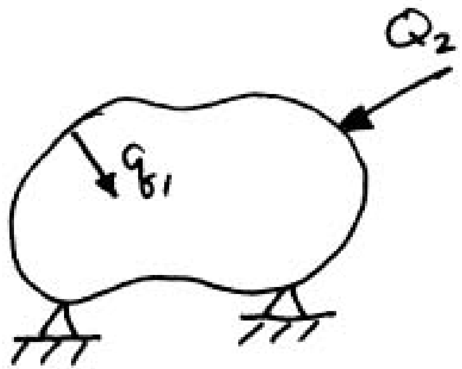
$$\underset{\sim}{q} = \underset{\sim}{C}\ \underset{\sim}{Q}$$

C_{ij} = <u>Flexibility Influence Coefficient</u>

and it gives the deflection at i due to a unit load at j

C_{12} = is deflection at 1 due to force at 2

Figure 21.2 **Representation of deflection point 1 due to load at point 2**



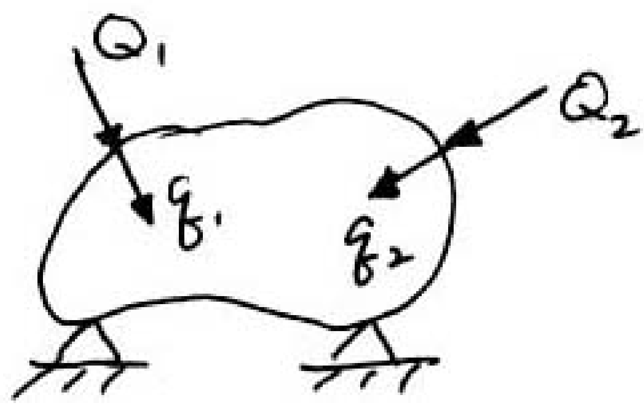
(<u>Note</u>: C_{ij} can mix types)

Very important theorem:

<u>Maxwell's Theorem of Reciprocal Deflection</u>

(Maxwell's Reciprocity Theorem)

Figure 21.3 **Representation of loads and deflections at two points on an elastic body**



q_1 due to unit load at 2 is <u>equal</u> <u>to</u> q_2 due to unit load at 1
i.e. $C_{12} = C_{21}$

Generally:

$$\boxed{C_{ij} = C_{ji}}$$

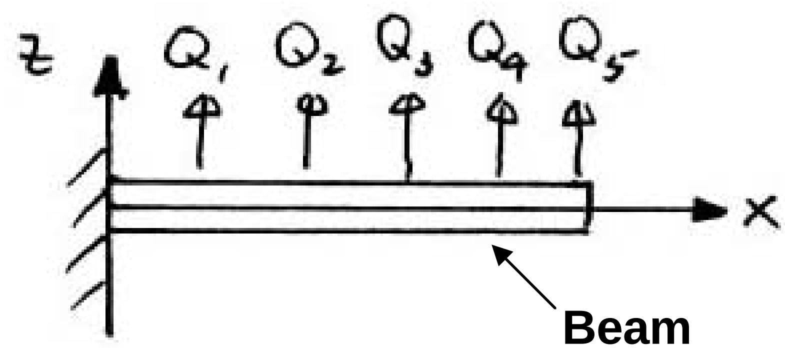
<u>symmetric</u>

This can be proven by energy considering (path independency of work)

--> Application of Flexibility Influence Coefficients

Look at a beam and consider 5 points…

Figure 21.4 **Representation of beam with loads at five points**



The deflections $q_1 \ldots q_5$ can be characterized by:

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & \cdots & \cdots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ C_{51} & \cdots & \cdots & \cdots & C_{55} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix}$$

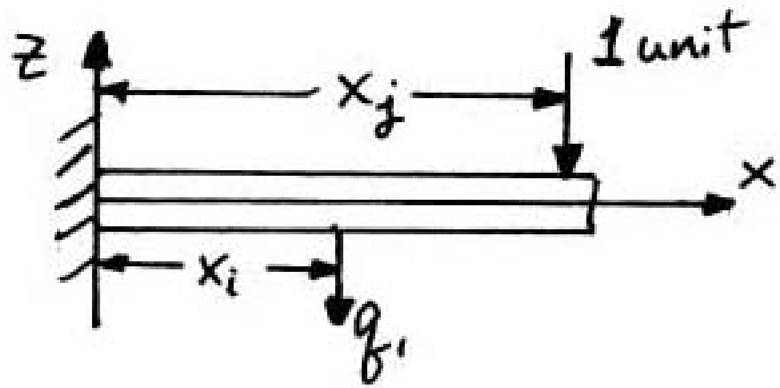
Since $C_{ij} = C_{ji}$, the $[C_{ij}]$ matrix is <u>symmetric</u>

Thus, although there are 25 elements to the C matrix in this case, only 15 need to be computed.

So, for the different loads $Q_1 \dots Q_5$, one can easily compute the $q_1 \dots q_5$ from previous work…

<u>Example</u>: C_{ij} for a <u>Cantilevered Beam</u>

Figure 21.5 **Representation of cantilevered beam under load**



find: C_{ij} --> deflection at i due to unit load at j

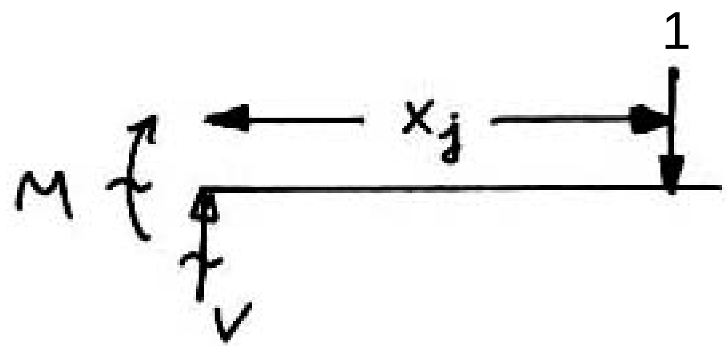
- Most efficient way to do this is via Principle of Virtual Work (energy technique)
- Resort here to using simple beam theory:

$$EI \frac{d^2 w}{dx^2} = M(x)$$

What is M(x)?

--> First find reactions:

Figure 21.6 **Free body diagram to determine reactions in cantilevered beam**

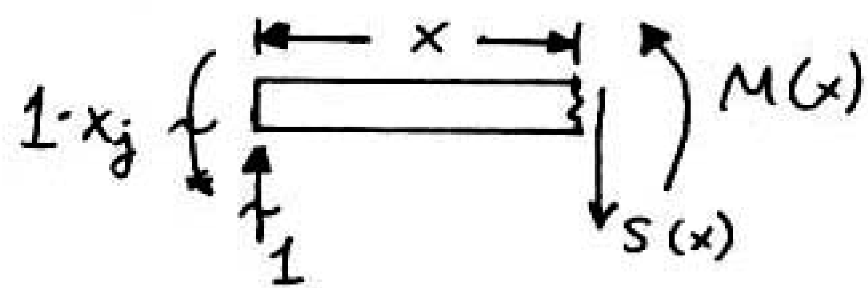


$\Rightarrow$ V = 1

M = -1x_j

--> Now find M(x). Cut beam short of x_j:

Figure 21.7 **Free body diagram to determine moment along cantilevered beam**



$$\sum M_x = 0 \quad +\circlearrowleft$$

$$\Rightarrow 1 \cdot x_j - 1 \cdot x + M(x) = 0$$

$$\Rightarrow M(x) = -1\left(x_j - x\right)$$

Plugging into deflection equation:

$$E I \frac{d^2 w}{dx^2} = -1\left(x_j - x\right)$$

for EI constant:

$$\frac{dw}{dx} = -\frac{1}{EI}\left(x_j x - \frac{x^2}{2}\right) + C_1$$

$$w = -\frac{1}{EI}\left(x_j \frac{x^2}{2} - \frac{x^3}{6}\right) + C_1 x + C_2$$

Boundary Conditions:

$$@\ x = 0 \quad w = 0 \Rightarrow C_2 = 0$$

$$@\ x = 0 \quad \frac{dw}{dx} = 0 \Rightarrow C_1 = 0$$

So:

$$w = -\frac{1}{EI}\left(x_j \frac{x^2}{2} - \frac{x^3}{6}\right)$$

evaluate at x_i:

$$w = \frac{1}{2EI}\left(\frac{x_i^3}{3} - x_i^2 x_j\right)$$

One important note:

w is defined as positive up, have defined q_i as positive down. So:

$$q_i = -w = \frac{1}{2EI}\left(x_i^2 x_j - \frac{x_i^3}{3}\right)$$

$$\Rightarrow \boxed{C_{ij} = \frac{1}{EI}\left(\frac{x_i^2 x_j}{2} - \frac{x_i^3}{6}\right)} \quad \text{for } x_i \leq x_j$$

Deflection, q_i, at x_i due to unit force, Q_j, at x_j

--> What about for $x_i \leq x_j$? Does one need to go through this whole procedure again?

No! Can use the same formulation due to the symmetry of C_{ij}

($C_{ij} = C_{ji}$)

--> Thus far have looked at the influence of a force on a displacement. May want to look at the "opposite": the influence of a displacement on a force. Do this via…

Stiffness Influence Coefficients

$\equiv k_{ij}$

where can write:

$$Q_1 = k_{11} q_1 + k_{12} q_2 + k_{13} q_3$$

$$Q_2 = k_{21} q_1 + k_{22} q_2 + k_{23} q_3$$

$$Q_3 = k_{31} q_1 + k_{32} q_2 + k_{33} q_3$$

or write:

$$\{Q_i\} = [k_{ij}]\{q_j\}$$

or:

$$\underset{\sim}{Q} = \underset{\sim}{k}\ \underset{\sim}{q}$$

If compare this with:

$$\underset{\sim}{q} = \underset{\sim}{C}\ \underset{\sim}{Q}$$

$$\Rightarrow \quad \underset{\sim}{k} = \underset{\sim}{C}^{-1}$$

$$[k_{ij}] = [C_{ij}]^{-1}$$ ← inverse matrix

Note: Had a similar situation in the continuum case:

$$\underset{\sim}{E} = \underset{\sim}{S}^{-1}$$

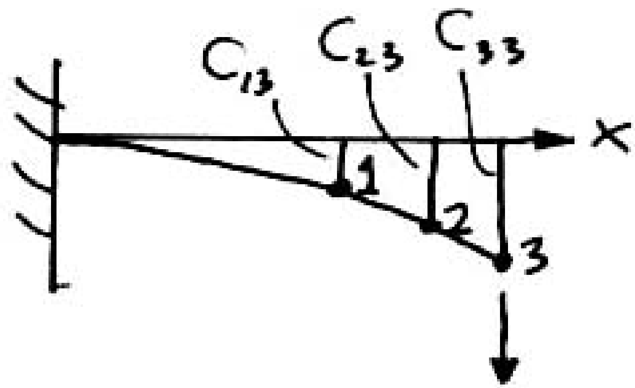
$\underset{\sim}{E}$: elasticity (stiffness)

$\underset{\sim}{S}$: compliance (flexibility)

--> Look at the Physical Interpretations:

Flexibility Influence Coefficients

Figure 21.8 **Physical representation of flexibility influence coefficients for cantilevered beam**



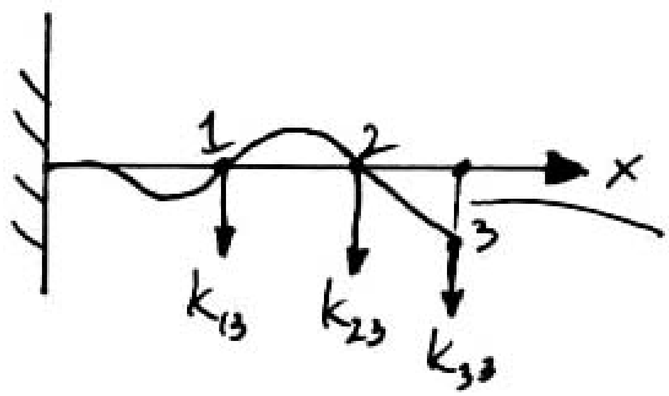
1 unit level

C_{ij} = displacement at i due to unit load at j

Note: This is only defined for **sufficiently constrained** structure

Stiffness Influence Coefficients

Figure 21.9 **Physical representation of stiffness influence coefficients for cantilevered beam**



k_{ij} = forces at i's to give a unit displacement at j and zero displacement everywhere else (at nodes)

(much harder to think of than C_{ij})

Note: This can be defined for unconstrained structures

--> Can find k_{ij} by:

- calculating $[C_{ij}]$ first, then inverting

$$k_{ij} = \frac{(-1)^{i+j}}{|\Delta|} \left| \begin{matrix} \text{minor of} \\ C_{ij} \end{matrix} \right|$$

($|\Delta|$ ← determinant of $[C_{ij}]$)

Note: k^{-1} may be singular (indicates "rigid body" modes)
--> rotation
--> translation
--> etc.

- calculating $[k_{ij}]$ directly from individual local $[k_{ij}]$ elements and adding up for the total system

  ↖ Most convenient way

Note: This latter method is the basis for finite element methods