Unit 20
Solutions for Single Spring-Mass Systems

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Return to the simplest system: the single spring-mass…

This is a one degree-of-freedom system with the governing equation:

$$m\ddot{q} + kq = F$$

First consider…

Free Vibration

⇒ Set F = 0

resulting in:

$$m\ddot{q} + kq = 0$$

The solution to this is the homogeneous solution to the general equation.

For an Ordinary Differential Equation of this form, know that the solution is of the form:

$$q(t) = e^{pt}$$

$$\Rightarrow \; mp^2 e^{pt} + k e^{pt} = 0$$

$$\Rightarrow \; mp^2 + k = 0$$ (in order to hold for all t)

$$\Rightarrow \quad p^2 = -\frac{k}{m}$$

$$\Rightarrow \quad p = \pm i\sqrt{\frac{k}{m}}$$

where:

$$i = \sqrt{-1}$$

$$\boxed{\sqrt{\frac{k}{m}} = \omega}$$

= natural frequency of single degree-of-freedom system

→ [rad/sec]

Important concept that natural frequency = $\sqrt{\frac{\text{stiffness}}{\text{mass}}}$

So have the equation:

$$q(t) = C_1 e^{+i\omega t} + C_2 e^{-i\omega t}$$

from mathematics, know this is:

$$\boxed{q(t) = C_1' \sin \omega t + C_2' \cos \omega t}$$ general solution

Now use the Initial Conditions:

$$@ \; t = 0 \quad q = q_0 \Rightarrow C_2' = q_0$$

$$@ \; t = 0 \quad \dot{q} = \dot{q}_0 \Rightarrow C_1' = \frac{\dot{q}_0}{\omega}$$

This results in:

$$q(t) = \frac{\dot{q}_0}{\omega} \sin \omega t + q_0 \cos \omega t$$

with:

$$\omega = \sqrt{\frac{k}{m}}$$

This is the basic, unforced response of the system

So if one gives the system an initial displacement A and then lets go:

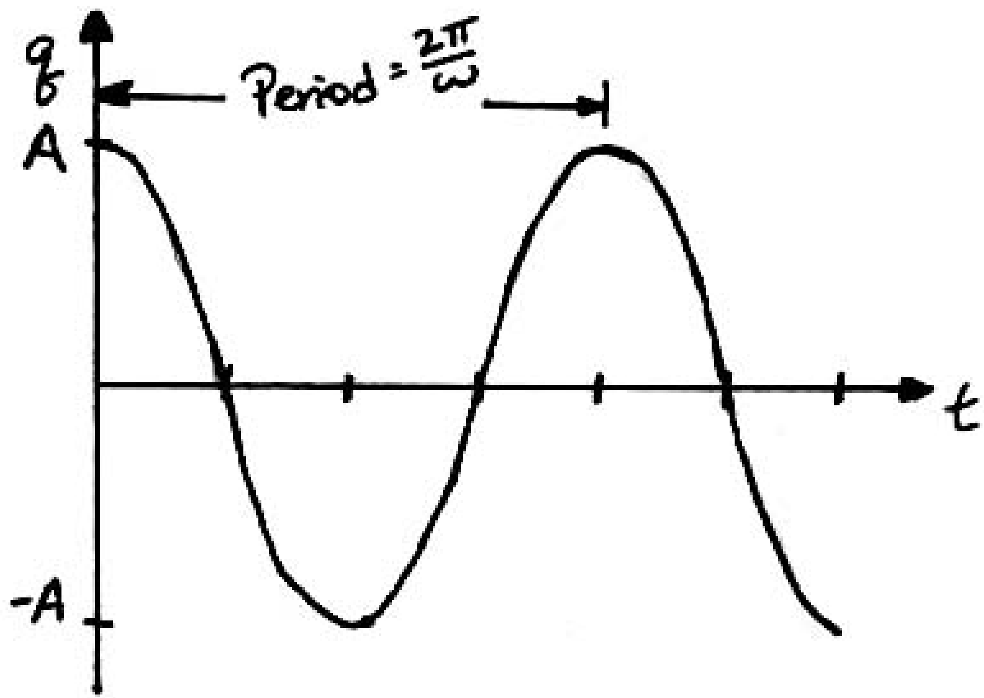
$$q_0 = A$$

$$\dot{q}_0 = 0$$

The response is:

$$q(t) = A\cos\omega t$$

Figure 20.1 **Basic unforced dynamic response of single spring-mass system**



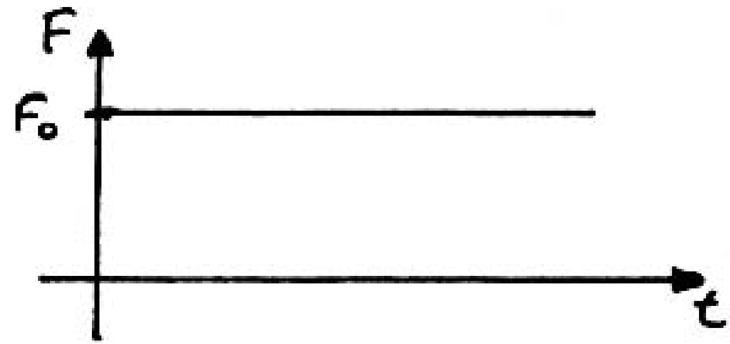
But, generally systems have a force, so need to consider:

Forced Vibration

The homogeneous solution is still valid, but must add a *particular solution*

The simplest case here is a constant load with time…

Figure 20.2 **Representation of constant applied load with time**



(think of the load applied suddenly ⇒ step function at t = 0)

The governing equation is:

$$m\ddot{q} + kq = F_0$$

The particular solution has no time dependence since the force has no time dependence:

$$q_{\text{particular}} = \frac{F_0}{k}$$

Now use the homogeneous solution with this to get the total solution:

$$q(t) = C_1 \sin \omega t + C_2 \cos \omega t + \frac{F_0}{k}$$

The Initial Conditions are:

$$@ \; t = 0 \quad q = 0$$
$$\dot{q} = 0$$

$$q(0) = 0 \Rightarrow C_2 = -\frac{F_0}{k}$$

$$\dot{q}(0) = 0 \Rightarrow C_1 = 0$$

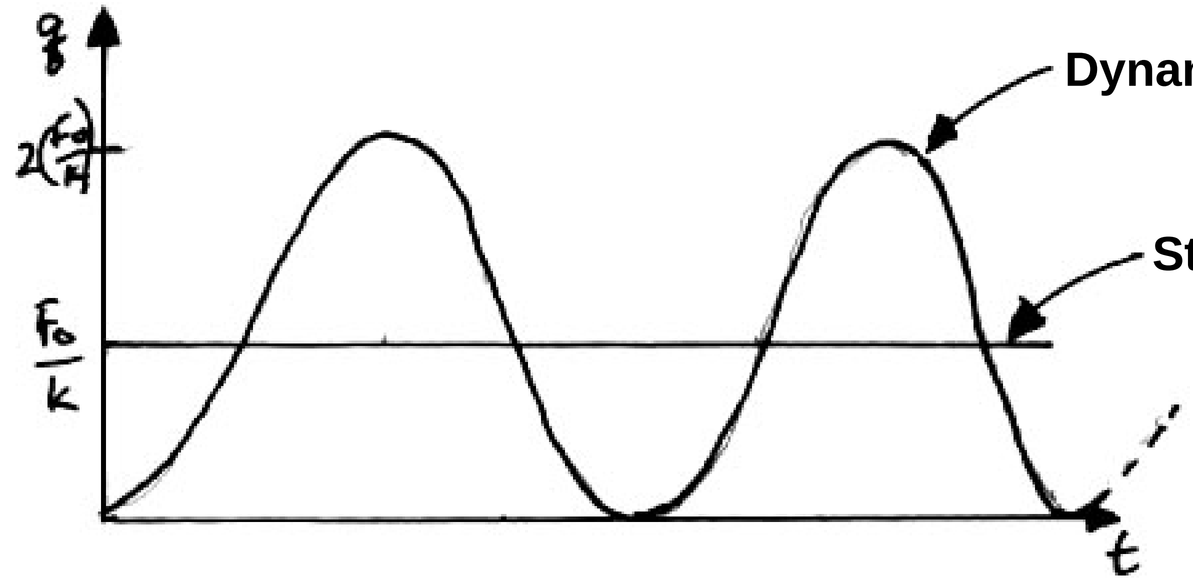
So the final solution is:

$$q = \frac{F_0}{k}\left(1 - \cos \omega t\right)$$

with $\omega = \sqrt{\frac{k}{m}}$

Plotting this:

Figure 20.3 **Ideal dynamic response of single spring-mass system to constant force**



Note that:

Dynamic response = 2 x static response

"dynamic magnification factor" - will be larger when considering stresses over their static values

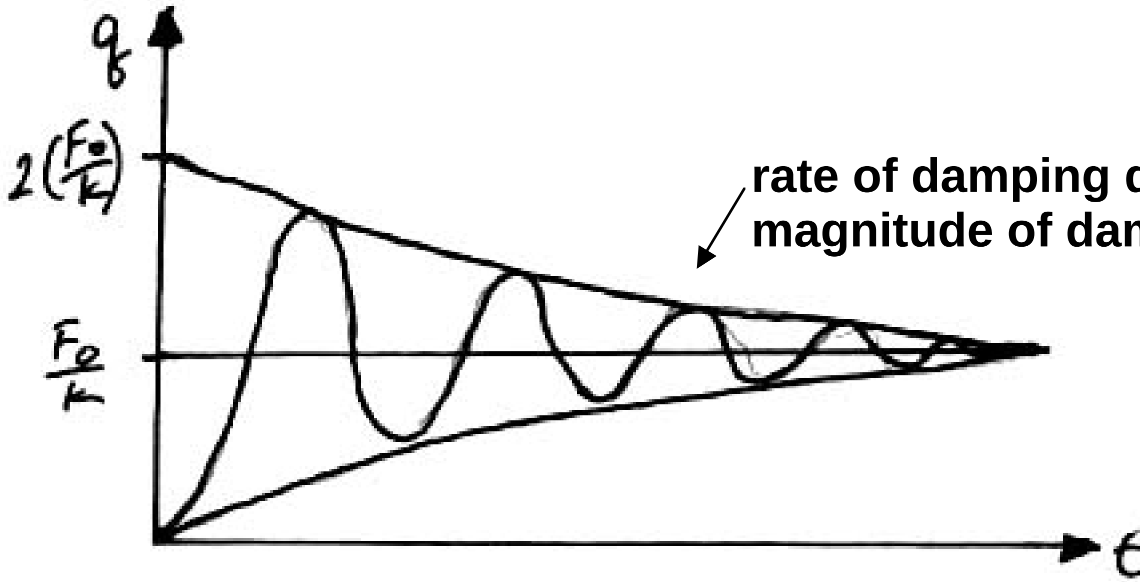
Know this doesn't really happen (i.e. response does not continue forever)

What has been left out?

DAMPING

Actual behavior would be…

Figure 20.4 **Actual dynamic response (with damping) of single spring-mass system to constant force**



rate of damping dependent on magnitude of damper (c)

Have considered a simple case. But, in general forces are not simple steps. Consider the next “level”…

<u>The Unit Impulse</u>

An impulse occurs at time t = τ. Such a force is represented by the

Dirac delta function: $F(t) = \delta(t - \tau)$

where:

$$\left.\begin{aligned} &\delta(t-\tau) = 0 \quad @ \quad t \neq \tau \\ &\delta(t-\tau) \to \infty \quad @ \quad t = \tau \\ &\int_{-\infty}^{\infty} \delta(t-\tau)\, dt = 1 \end{aligned}\right\}$$

and:

Recall: force x time = impulse

Hence,

$$\int_{-\infty}^{\infty} g(t) \underbrace{\delta(t-\tau)}_{= 0 \text{ everywhere but at } t = \tau} dt$$

$$= g(\tau) \underbrace{\int_{-\infty}^{\infty} \delta(t-\tau)\, dt}_{= 1}$$

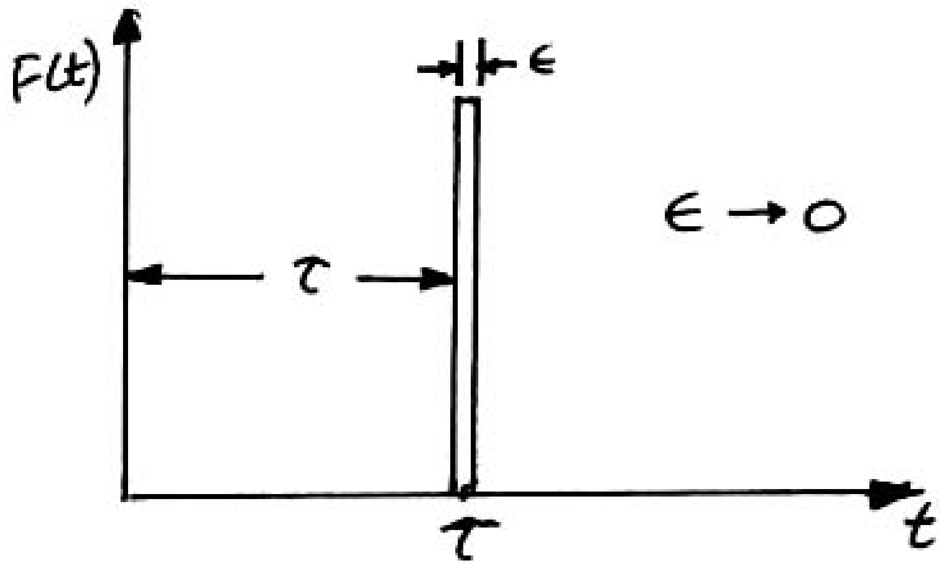
now a constant with regard to time (refers to $g(\tau)$)

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} g(t)\, \delta(t - \tau)\, dt = g(\tau)}$$

Use this force function in the governing equation:

$$m\ddot{q} + kq = \delta(t - \tau)$$

Figure 20.5 **Representation of unit impulse force at time τ**



Use Initial Condition of system at rest:

$$@\ t < \tau \quad q = 0$$

$$\dot{q} = 0$$

To get the result, integrate the equation over the regime where the force is nonzero (t = τ to τ + ε)

$$\Rightarrow \int_{\tau}^{\tau+\varepsilon} m \frac{d^2 q}{dt^2}\, dt + \underbrace{k \int_{\tau}^{\tau+\varepsilon} q\, dt}_{\text{use trapezoidal rule}} = \int_{\tau}^{\tau+\varepsilon} \delta(t-\tau)\, dt$$

<u>Trapezoidal</u> <u>Rule</u>:

$$\int_A^B q\, dt = \frac{1}{2}\left[q(B) + q(A)\right] \times (t_B - t_A)$$

This gives:

$$\Rightarrow m\left[\left(\frac{dq}{dt}\right)_{\tau+\varepsilon} - \underbrace{\left(\frac{dq}{dt}\right)_{\tau}}_{\text{= 0 from Initial Condition}}\right] + k\frac{1}{2}\left[q(\tau+\varepsilon) + \underbrace{q(\tau)}_{\text{= 0 from Initial Condition}}\right]\varepsilon = 1$$

(ε → goes to zero since $\varepsilon \to 0$)

All this results in:

$$\left(\frac{dq}{dt}\right)_{\tau+\varepsilon} = \frac{1}{m}$$

$$(q)_{\tau+\varepsilon} = 0$$

A unit impulse at time τ is thus a free vibration problem with an initial velocity (equal in magnitude to the inverse of the mass)

Note: Units are consistent since the integral of δ(t - τ) is a force x time. So:

$$\left[\frac{dq}{dt}\right] = \frac{\text{Force} \times \text{time}}{\text{mass}} = \frac{\text{mass} \times \frac{\text{length}}{(\text{time})^2} \times \text{time}}{\text{mass}}$$

$$= \frac{\text{length}}{\text{time}} = [\text{velocity}]$$

So use the homogeneous solution:

$$q(t) = C_1 \sin \omega t + C_2 \cos \omega t$$

but:

- shift the time {t' = 0 @ t = τ ⇒ (t - τ)}
- use:

$$\dot{q}(t = \tau) = \frac{1}{m}$$

$$q(t = \tau) = 0$$

Thus:

$$q(t) = C_1 \sin\omega(t - \tau) + C_2 \cos\omega(t - \tau)$$

Initial Conditions give:

$$q(t = \tau) = 0 \Rightarrow C_2 = 0$$

$$\dot{q}(t = \tau) = \frac{1}{m} \Rightarrow \frac{1}{m} = C_1 \omega \Rightarrow C_1 = \frac{1}{m\,\omega}$$

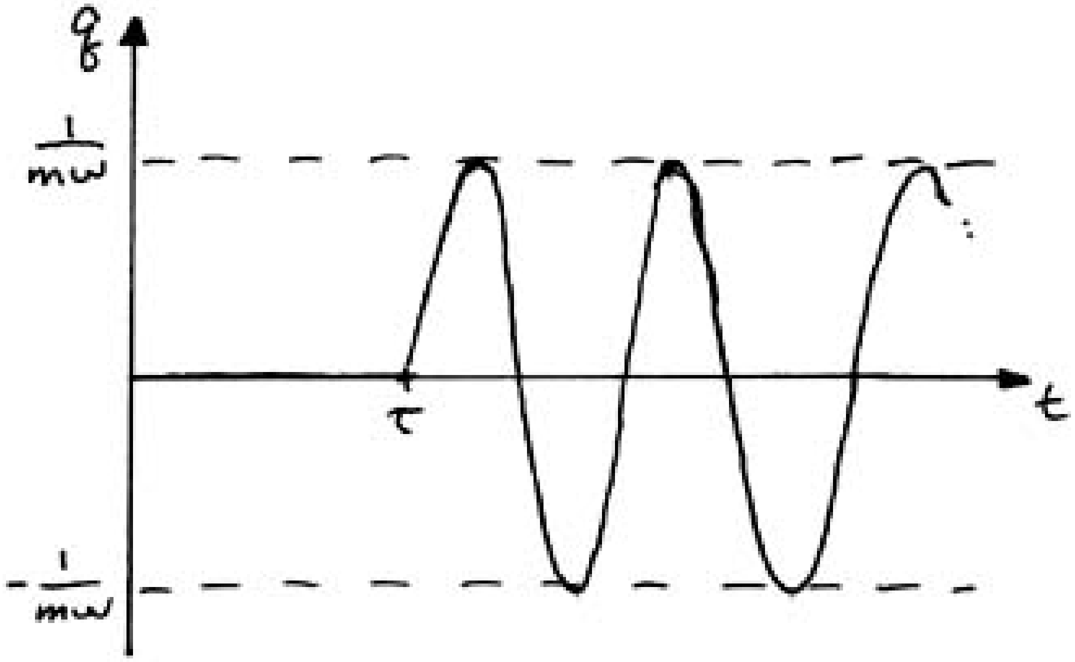
Thus:

$$q(t) = \frac{1}{m\,\omega} \sin\omega\,(t - \tau)$$

with:

$$\omega = \sqrt{\frac{k}{m}}$$

Figure 20.6 **Dynamic response of single spring-mass system to unit force impulse at time τ**



Summarizing…the response to a unit impulse at t = τ is:

$$q(t) = \begin{cases} \frac{1}{m\,\omega} \sin \omega \left(t - \tau\right) & \text{for } t \geq \tau \\ 0 & \text{for } t \leq \tau \end{cases}$$

For an impulse of arbitrary magnitude I_o, multiply the solution by I_o

For convenience, write this particular response as:

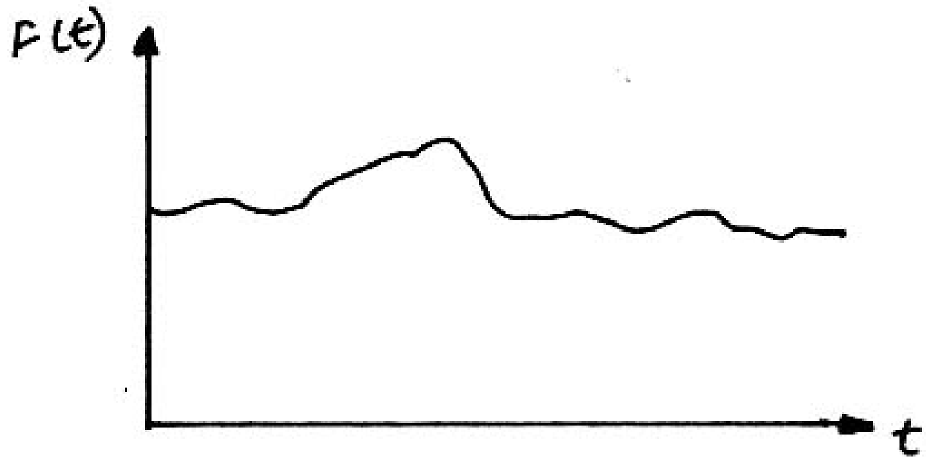
$h(t - \tau)$ = response to unit impulse

Now, how does one progress to an…

Arbitrary Force

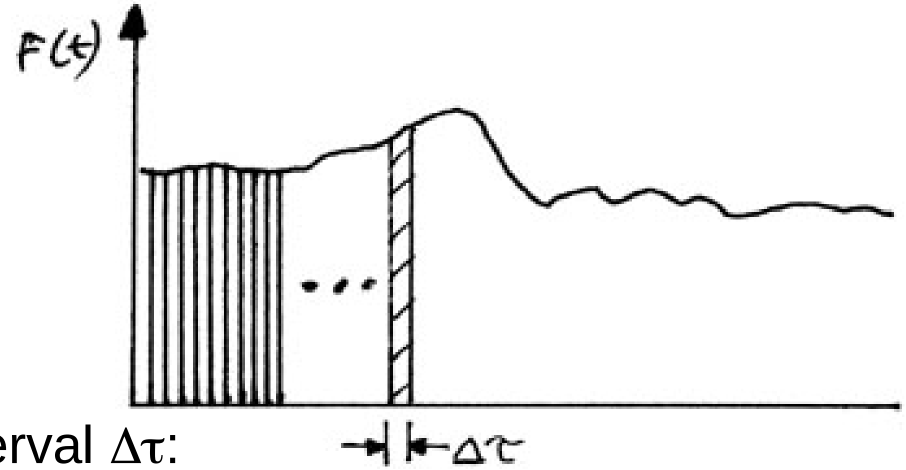
Consider some arbitrary force with time:

Figure 20.7 **Representation of arbitrary force with time**



Break the force into/represent it as a series of impulses:

Figure 20.8 **Representation of arbitrary force with time as series of impulses**



over any interval Δτ:

Impulse = F (τ) δ (t - τ)

The response to any particular impulse is:

$$q(t) = F(\tau)\,\Delta\tau\; h(t - \tau)$$

where h(t - τ) is characteristic response of system to an impulse.

So the total response is the summation of the responses to all the impulses. In the limit, there are infinite impulses leading to the integral:

$$q(t) = \int_0^t F(\tau)\, h(t - \tau)\, d\tau$$

This is known as:

Duhamel's integral or
The convolution integral or
The linear superposition integral

This is a general case. For the particular single spring-mass system:

$$q(t) = \frac{1}{m\,\omega}\int_0^t F(\tau)\,\sin\omega\,(t-\tau)\,d\tau$$

response to arbitrary F(t)

Additional initial conditions (velocity and displacement) can be added to this through the homogeneous solution to give:

$$q(t) = \frac{1}{m\,\omega}\int_0^t F(\tau)\,\sin\omega\,(t-\tau)\,d\tau + \frac{\dot{q}(0)}{\omega}\sin\omega t + q(0)\cos\omega t$$

Of all the arbitrary forces, there is one form of particular interest:

Sinuisoidal Force

This is an important basic input

- motors
- helicopters
- other cyclical cases

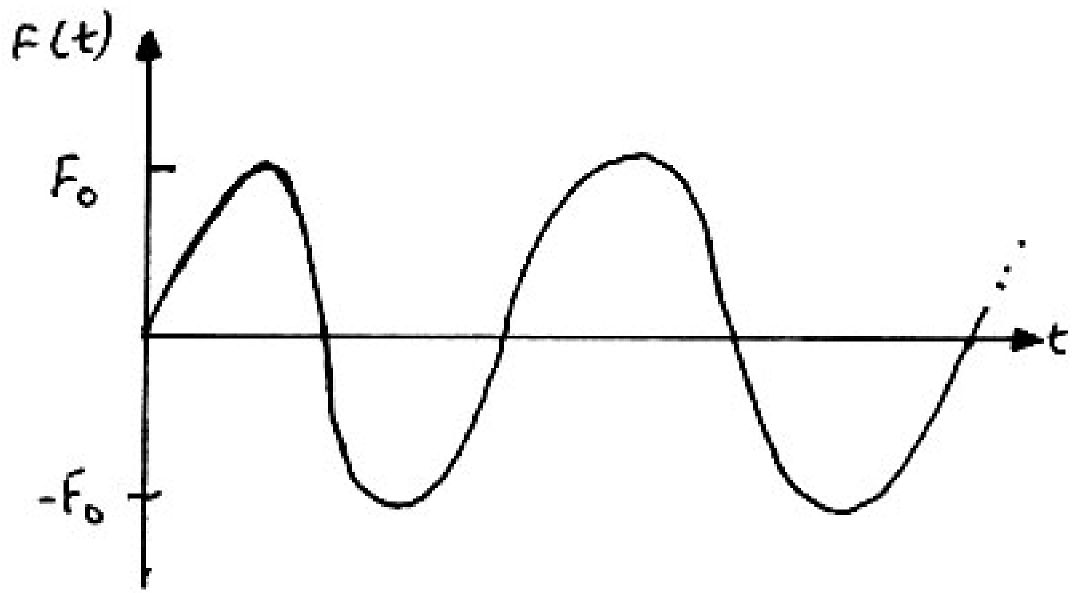
Use the integral previously developed for the case of the basic force:

$$F(t) = F_o \sin \Omega t$$

(use capital Ω to differentiate from system natural frequency)

Ω - forcing frequency
ω - system (response) natural frequency

Figure 20.9 **Representation of sinusoidal forcing function**



The solution is thus:

$$q(t) = \frac{F_0}{m\omega} \int_0^t \sin \Omega\tau \sin \omega (t - \tau) d\tau$$

One can perform this integral to get the solution

<u>or</u>

Go back to the original Ordinary Differential Equation:

$$m\ddot{q} + kq = F_o \sin \Omega t$$

The overall solution is:

$$q(t) = \underbrace{C_1 \sin \omega t + C_2 \cos \omega t}_{\text{homogeneous solution}} + q_{\text{particular}}$$

Can see:

$$q_{\text{particular}} = C_3 \sin \Omega t$$

Plugging this in the governing Ordinary Differential Equation:

$$- m \Omega^2 C_3 \sin \Omega t + k\, C_3 \sin \Omega t = F_o \sin \Omega t$$

$$\Rightarrow \ C_3 = \frac{F_o}{- m \Omega^2 + k} = \frac{F_o}{k\left(1 - \Omega^2 \dfrac{m}{k}\right)}$$

use:

$$\omega = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad C_3 = \frac{F_o}{k\left(1 - \dfrac{\Omega^2}{\omega^2}\right)}$$

This gives the overall solution of:

$$q(t) = \underbrace{C_1 \sin \omega t + C_2 \cos \omega t}_{\text{Starting transient with natural frequency (this will eventually die out due to damping)}} + \underbrace{\frac{F_o}{k\left(1 - \frac{\Omega^2}{\omega^2}\right)} \sin \Omega t}_{\text{Steady state response following frequency of forcing function}}$$

Now, using the Initial Conditions:

$q(0) = 0 \Rightarrow C_2 = 0$

$$\dot{q}(0) = 0 \Rightarrow \omega C_1 + \frac{F_o \Omega}{k\left(1 - \frac{\Omega^2}{\omega^2}\right)} = 0$$

$$\Rightarrow C_1 = - \frac{F_o \frac{\Omega}{\omega}}{k\left(1 - \frac{\Omega^2}{\omega^2}\right)}$$

Thus, the final solution is:

$$q(t) = \frac{F_o}{k\left(1 - \frac{\Omega^2}{\omega^2}\right)} \left(\underbrace{- \frac{\Omega}{\omega} \sin \omega t}_{\text{starting transient}} + \underbrace{\sin \Omega t}_{\text{steady state}} \right)$$

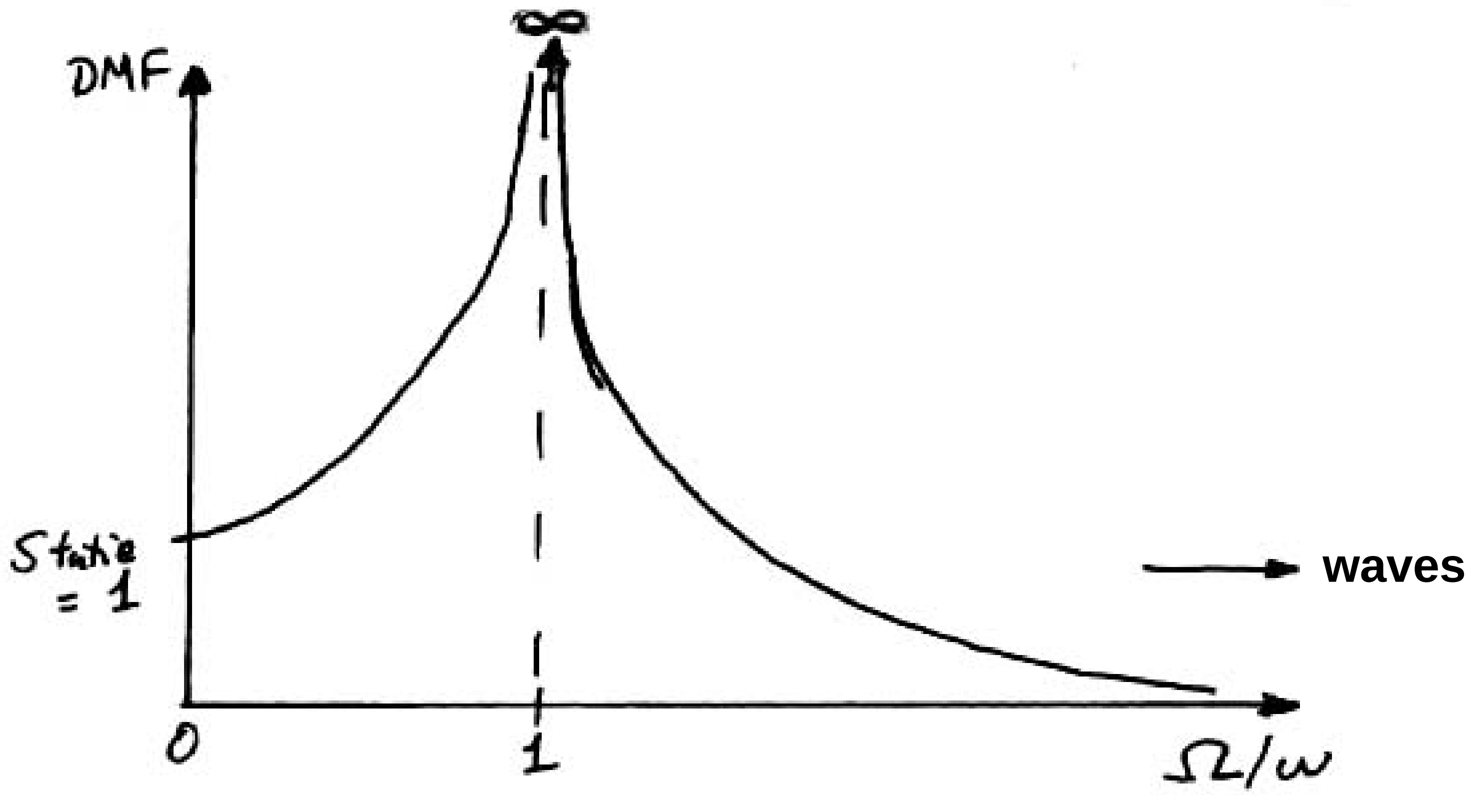
Notes:

- $\frac{F_o}{k}$ = static response

- $\frac{1}{\left(1 - \frac{\Omega^2}{\omega^2}\right)}$ = Dynamic Magnification Factor (DMF)

- For low Ω, response approximately static
- For high Ω, response goes to zero *(waves!)*
- For medium Ω….DMF varies. Medium depends on ω which is system structural response
- At $\omega = \Omega$, DMF $\rightarrow \infty$ This is known as "Resonance"

Figure 20.10 **Dynamic Magnification Factor versus ratio of frequency of forcing function to system natural frequency**



Thus, if one excites a system at/near its natural frequency, very large responses result (damping and nonlinearities keep the response from going to infinity)

Examples: Flutter (window blinds in wind!)

Rule of Thumb: stay away from natural frequency(ies) of system.

Further Note: an arbitrary force can be broken down into a sinusoidal series. If any of the force components of importance have a frequency near a system natural frequency $\Rightarrow$ Trouble!

The next task is to take the basic points learned for this single degree-of-function system and extend them to a multiple degree-of-function system.

But first need to see how one can accomplish this…