Unit 18
Other Issues In Buckling/Structural Instability

Readings:

Rivello	14.3, 14.5, 14.6, 14.7 (read these at least, others at your "leisure")
	Ch. 15, Ch. 16
Timoshenko	Theory of Elastic Stability
Jones	Mechanics of Composite Materials, Ch. 5

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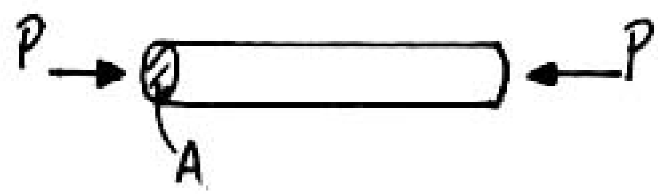
Have dealt, thus far, with perfect columns, loading eccentricities, and beam-columns. There are, however, many more issues in buckling/(static) structural instability, most of which will try to touch on.

(a) Buckling versus Fracture

Have looked at columns that are long enough such that they buckle. However, it is possible that the material *compressive* ultimate stress may be reached before the static instability occurs.

Consider short/"squat" column (saw in Unified)

Figure 18.1 **Representation of short column under compressive load**



$$\sigma = \frac{P}{A}$$

If $\sigma = \sigma_{compressive\ ultimate}$ before $P = P_{cr}$, then failure occurs by material failure in compression

"squashing"

Using the "slenderness ratio" previously defined:

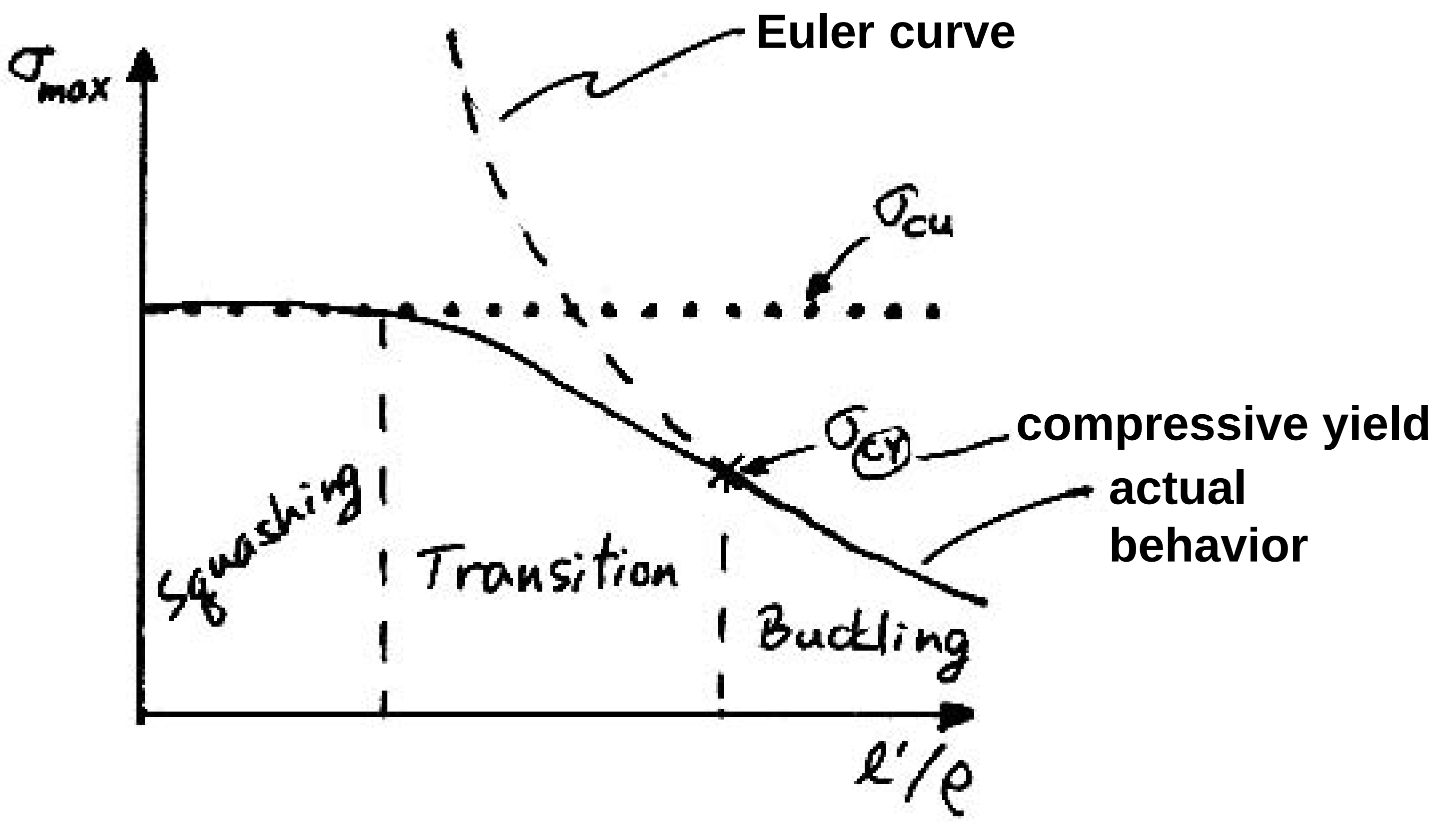
$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{\left(l'/\rho\right)^2}$$

where:

$$l' = \frac{l}{\sqrt{c}}$$

"define" a column by its slenderness ratio and can plot the behavior and "failure mode" of various columns…

Figure 18.2 **Summary plot showing general behavior of columns based on stress level versus slenderness ratio**

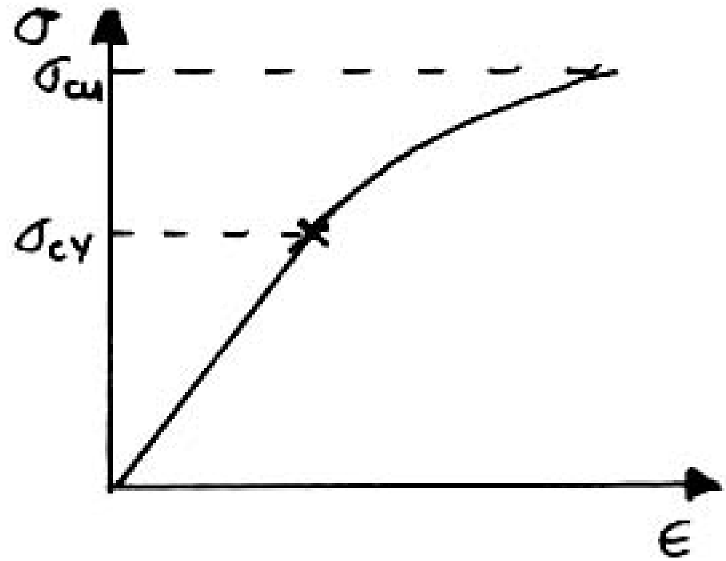


Regions of values depend on E and σ_{cu}

What happens in the transition region?

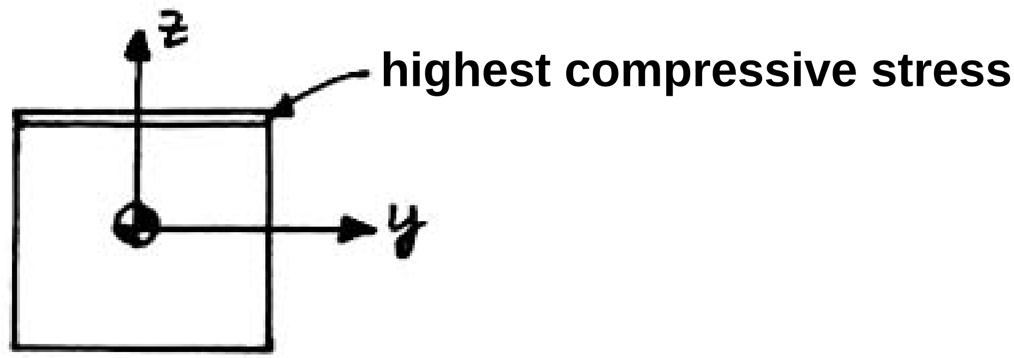
(b) Progressive Yielding

Figure 18.3 **Typical stress-strain plot for a ductile metal (in compression)**



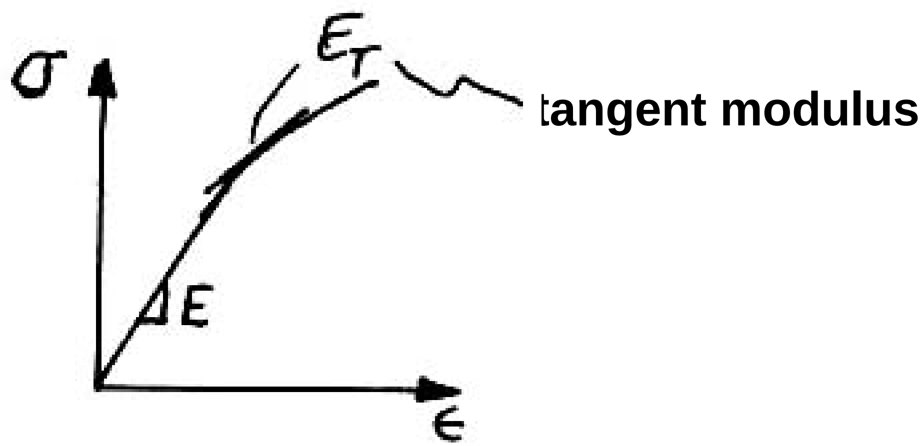
As the column is loaded, there is some deflection due to slight imperfections. This means the highest load is at the outer part of the beam-column.

Figure 18.4 **Representation of region of highest stress in cross-section of beam-column**



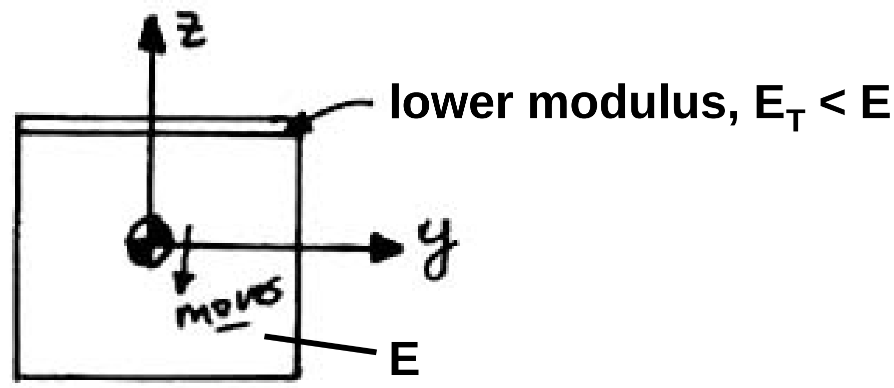
Thus, this outer part is the first part to yield. As the material yields, the modulus decreases.

Figure 18.5 **Representation of tangent modulus**



This changes the location of the centroid…

Figure 18.6 **Representation of change in location of centroid of cross-section due to local yielding**



This continues and it may eventually “squash” or buckle (or a combination)

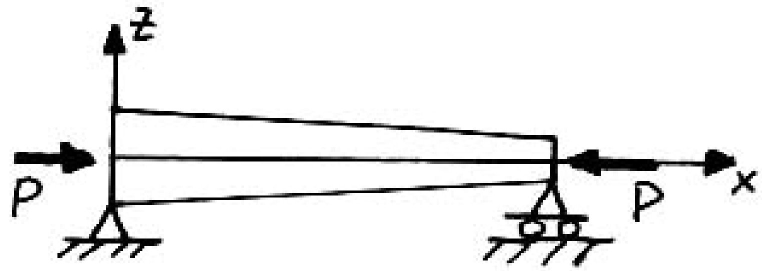
--> *See Rivello 14.6*

(c) Nonuniform Beam-Columns

Have looked only at beams with uniform cross-sectional property EI. Now let this vary with x (most likely I, not E).

Example: Tapered section

Figure 18.7 **Representation of beam-column with tapered cross-section**



EI = EI(x)

Thus, the governing equation is:

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2}\right) + P\,\frac{d^2w}{dx^2} = 0$$

must keep this "inside" the derivative

Solve this via *numerical techniques*:

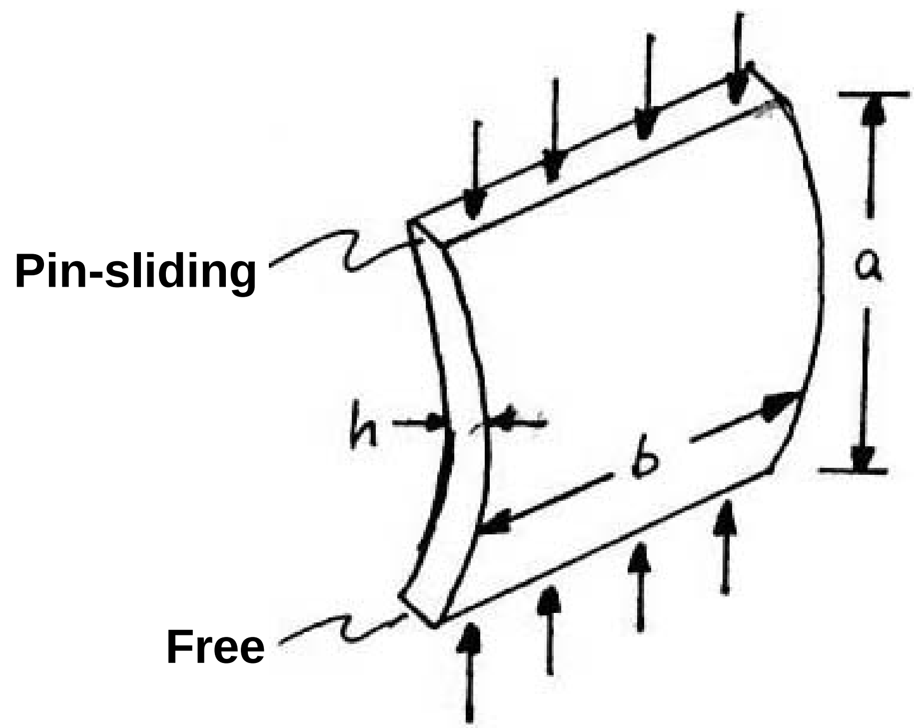
- Energy Methods
- Galerkin
- Finite Element Method
- Finite Difference
- Rayleigh-Ritz

--> See Rivello 14.3

(d) Buckling of Plates

Thus far have considered a "one-dimensional" problem (structural property of main importance is ℓ, besides EI). Now have a two-dimensional structure (a "*plate*"):

Figure 18.8 **Representation of plate under compressive load**



The Poisson's ratio enters into play here. For an isotropic plate get:

$$P_{cr} = \frac{\pi^2 E I}{l^2 \left(1 - \nu^2\right)}$$

where: $\begin{cases} \ell = a \\ I = 1/12\ bh^3 \end{cases}$

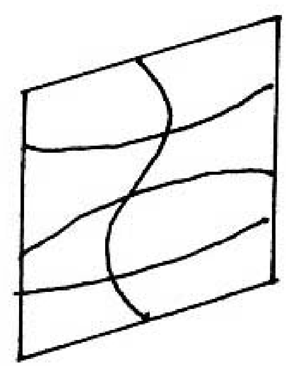
$$\Rightarrow \sigma_{cr} = \frac{P_{cr}}{bh} = \frac{\pi^2}{12} \frac{E}{1 - \nu^2} \left(\frac{h}{a}\right)^2$$

whereas the column buckling load is

$$P_{cr} = \frac{\pi^2 E\, I}{\ell^2} = \frac{\pi^2 E\, A\, h^2}{\ell^2 12} \Rightarrow \sigma_{cr} = \frac{\pi^2 E}{12} \left(\frac{h}{\ell}\right)^2$$

The buckled shape will have components in both directions:

Figure 18.9 **Representation of deflection of buckled square plate with all sides simply-suppo**



$$w = w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

--> can have contributions of many modes. Will depend on boundary conditions on all four sides.

--> See Rivello, Ch. 15

Even more complicated for orthotropic plates as the modulus varies in the two directions.

(must also look at buckling due to shear loads)

--> See Jones, Ch. 5

Note: for some "weird" anisotropic plates with shear couplings, can get buckling under tension.

(**Key question**: Is there an induced compressive stress in some direction?)

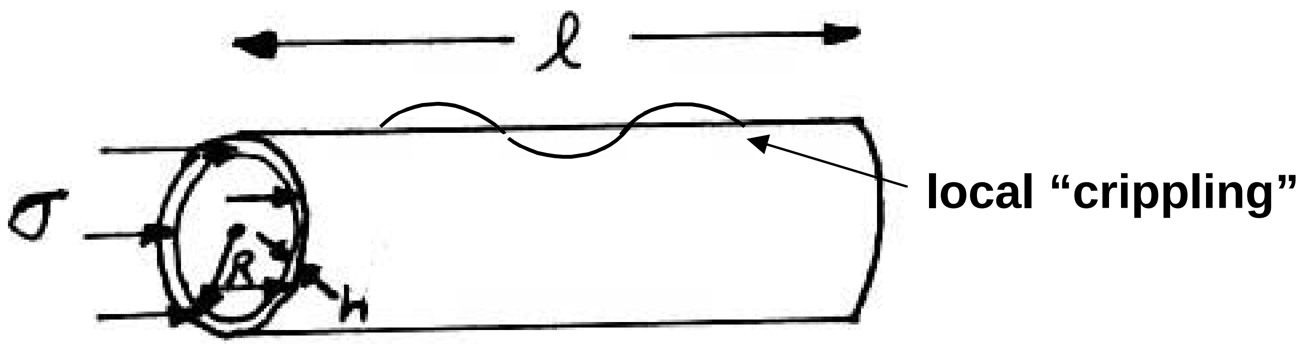
→ think back to basic definition of instability…

(e) Cylinders

("thin-walled things", like shells)

Have dealt with "*global*" instabilities. However, buckling can also be a "*local*" instability.

Figure 18.10 **Representation of crippling in thin cylinder under axial compressive load**



total axial load:

$$P = \sigma \, (2\pi) \, Rh$$

for an isotropic cylinder:

$$\sigma_{cr_{(linear)}} = 0.606 \, E \, \frac{h}{R}$$

The actual load where the local instability sets in is less than that predicted from linear theory due to imperfections in both geometry and loading:

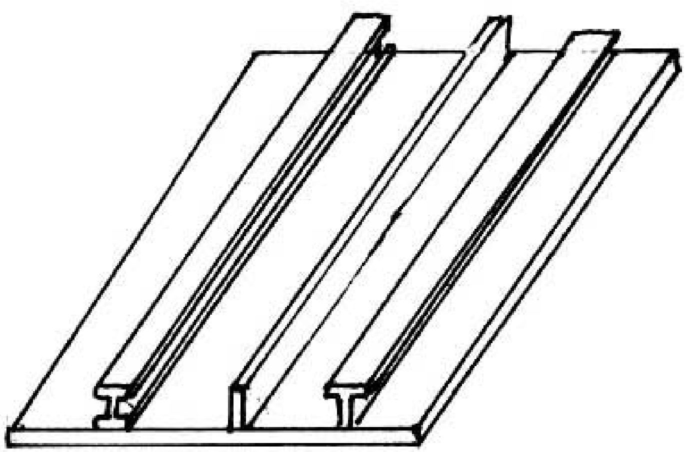
$$\sigma_{cr_{(actual)}} \approx (0.15 \text{ to } 0.9)\, \sigma_{cr_{(linear)}}$$

--> *See Rivello, Ch. 15*

(f) Reinforced Plates

A common design in aerospace structures (and many other structures) is to reinforce a plate with stiffeners:

Figure 18.11 **Representation of plate with stiffeners**



The buckling can take place at several levels

- buckling of panels between stiffness
- buckling of "parts" of stiffeners (e.g., flange, web)
- global instability

This can occur on a progressive basis.

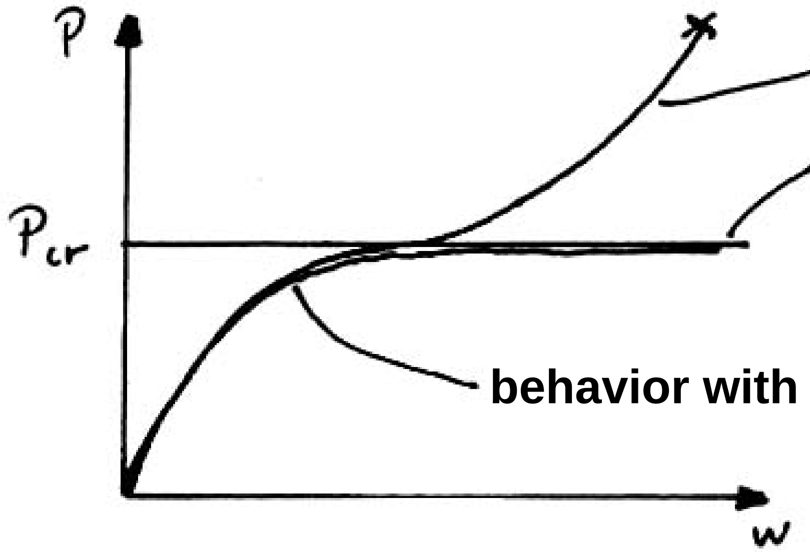
Analysis often uses only stiffness as carrying the load for buckling (axial load) or talk about "effective width" of skin

(this was previously discussed in talking about general shell beams and holds true for buckling)

(g) Post-buckling

When talked about buckling, talked about bifurcation. In that case w --> ∞ as P --> P_{cr} (with imperfections). In reality, a structure can carry load after buckling ("*post-buckling*" behavior).

Figure 18.12 **Representation of post-buckling behavior via load-deflection plot**



The critical assumption which breaks down is:

SMALL DEFORMATIONS

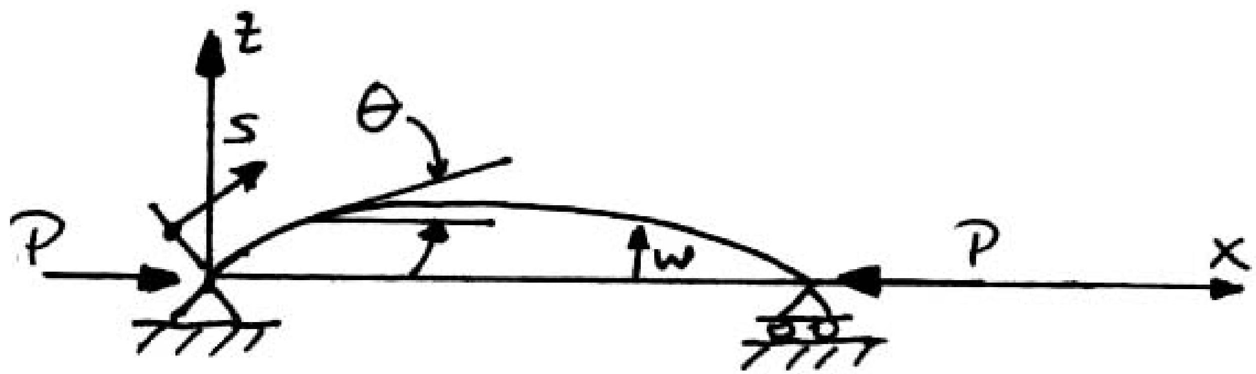
Must now account for geometrical nonlinear effects.

(Note: material nonlinear effects will also enter in as approach σ_{cu})

Consider: Post-Buckling of a beam-column

(the issues are the same for a plate)

Figure 18.13 **Representation of post-buckling of a beam-column**



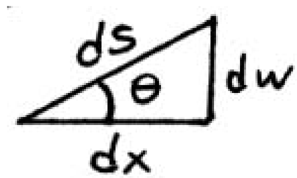
For large deflections, the moment-curvature equation is:

$$M(s) = EI\,\frac{d\theta}{ds}$$

where:

$$\frac{d\theta}{ds} = \text{curvature}$$

Look at a beam section:



$$\frac{dw}{ds} = \sin\theta$$

differentiating:

$$\frac{d^2w}{ds^2} = \cos\theta \frac{d\theta}{ds}$$

or:

$$\frac{d\theta}{ds} = \frac{1}{\cos\theta} \frac{d^2w}{ds^2}$$

with:

$$\cos\theta = \frac{dx}{ds} = \frac{\sqrt{ds^2 - dw^2}}{ds} = \sqrt{1 - \left(\frac{dw}{ds}\right)^2}$$

or:

$$\cos\theta = \sqrt{1 - \sin^2\theta}$$

So:

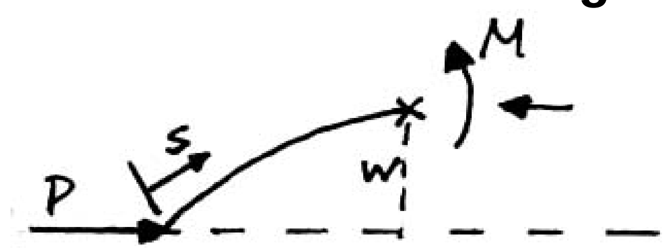
$$\text{curvature} = \frac{d\theta}{ds} = \frac{1}{\sqrt{1 - \left(\frac{dw}{ds}\right)^2}} \frac{d^2 w}{ds^2}$$

For "moderate" angles θ: (via expansion)

$$\frac{d\theta}{ds} = \left[1 + \underbrace{\frac{1}{2}\left(\frac{dw}{ds}\right)^2}_{\text{nonlinear term}} + \text{H.O.T.}\right] \frac{d^2 w}{ds^2}$$

In the absence of any primary moment, the moment at any point s is due to the deflection w at that point:

Figure 18.14 **Representation of resultants along the beam-column**



$$+\circlearrowleft \quad \sum M = 0$$

$\Rightarrow$ M + Pw = 0 $\quad\quad \Rightarrow$ M = -Pw

So the basic Post-Buckling equation becomes:

$$\left[1 + \frac{1}{2}\left(\frac{dw}{ds}\right)^2 + \text{H.O.T.}\right]\frac{d^2w}{ds^2} + \frac{P}{EI}w = 0$$

This can be solved via

- Numerical Techniques
- Energy Method

--> Effect appears to "stiffen" the behavior

Consider one (numerical) technique known as the…

Galerkin Method

1. Assume a mode that satisfies the boundary conditions

$$w = \underbrace{q_1}_{\text{unknown coefficient}} \underbrace{\sin \pi s/\ell}_{\text{assumed mode shape satisfies all boundary conditions}}$$

2. Integrate a weighted average of the solution and the Ordinary Differential Equation

(Are minimizing the "residuals")

$$\int_0^l \underbrace{q_1 \sin \frac{\pi s}{l}}_{\text{assumed mode shape}} \underbrace{\begin{Bmatrix}\text{Differential} \\ \text{equation}\end{Bmatrix}}\, ds = 0$$

assumed mode shape

Here: $\left[1 + \frac{1}{2}\left(\frac{dw}{ds}\right)^2\right] \frac{d^2 w}{ds^2} + \frac{P}{EI} w$

This gives:

$$-q_1 \frac{\pi^2}{2l} - q_1^{\,3} \frac{\pi^4}{16 l^3} + q_1 \frac{Pl}{2EI} = 0$$

Solving:

$$q_1 \left[\frac{\pi^4}{16 l^3} q_1^{\,2} + \frac{\pi^2}{2l} - \frac{Pl}{2EI}\right] = 0$$

Get:

- $q_1 = 0$ (trivial solution: w = 0)
- $q_1^2 = \frac{8l^2}{\pi^2}\left[\frac{Pl^2}{\pi^2 EI} - 1\right]$

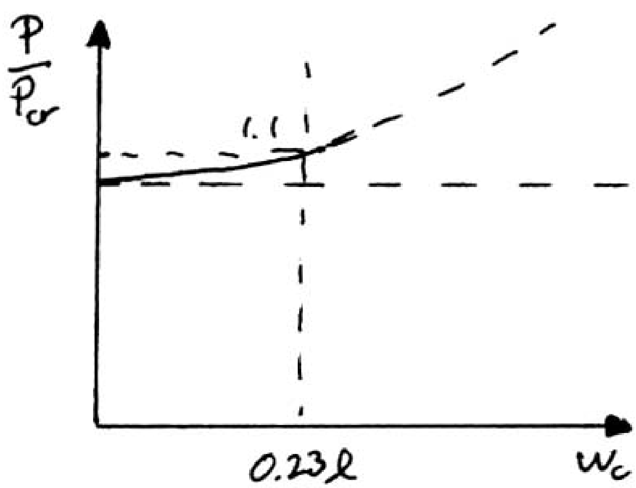
this latter gives:

$$q_1 = 0.903\, l \sqrt{\frac{P}{P_{cr}} - 1}$$

Note: q_1 only for $P/P_{cr} > 1$

Plotting P/P_{cr} vs. w_c (= q_1) gives:

Figure 18.15 **Representation of load versus center deflection for post-buckled beam-column based on Galerkin Method**



For w_c/ℓ > 0.3, include more terms:

$$1 + \frac{1}{2}\left(\frac{dw}{ds}\right)^2 + \frac{3}{8}\left(\frac{dw}{ds}\right)^4 + \ldots$$

<u>See</u>: Rivello, Timoshenko & Gere

<u>--> Postscript: Buckling and Failure</u>

When is a structure that buckles considered to have failed?

- Recall discussion of "failure" at beginning of term.
- There is not (physically…only mathematically) "a" point of buckling. What happens is that deflection increases with load very rapidly.
- If failure is deflection-based, look at deflection; if stress/strain-base, look at that…

Must consider:

- ***pre**-buckling behavior:* imperfections cause deflections and stresses. These may cause failure before "buckling"
- ***post**-buckling behavior:* "extra" stiffening at large deflections may result in ability to carry deflections and stresses such that failure is after "buckling"