Unit 17
The Beam-Column

Readings:

Theory of Elastic Stability, Timoshenko (and Gere), McGraw-Hill, 1961 (2nd edition), Ch. 1

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Thus far have considered separately:

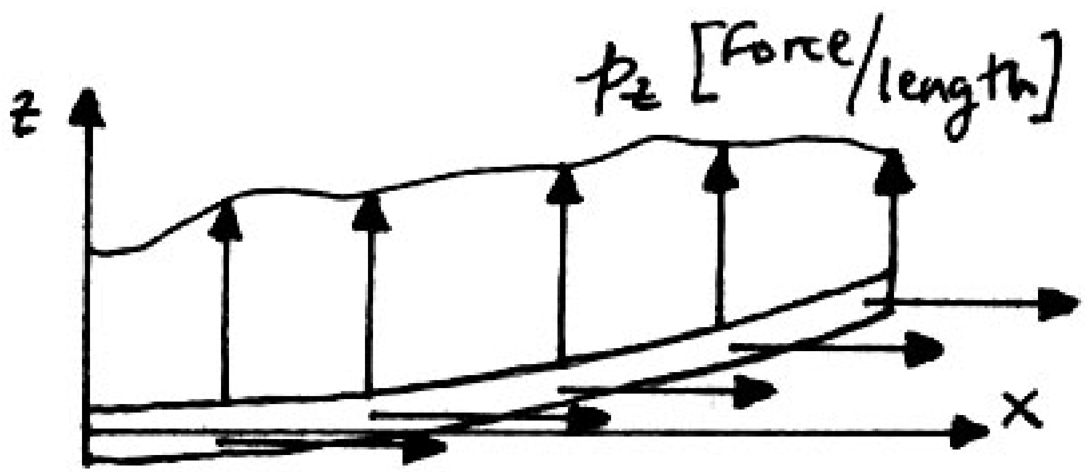
- beam -- takes bending loads
- column -- takes axial loads

Now combine the two and look at the "*beam-column*"

(Note: same geometrical restrictions as on others:
ℓ >> cross- sectional dimensions)

Consider a beam with an axial load (general case):

Figure 17.1 **Representation of beam-column**

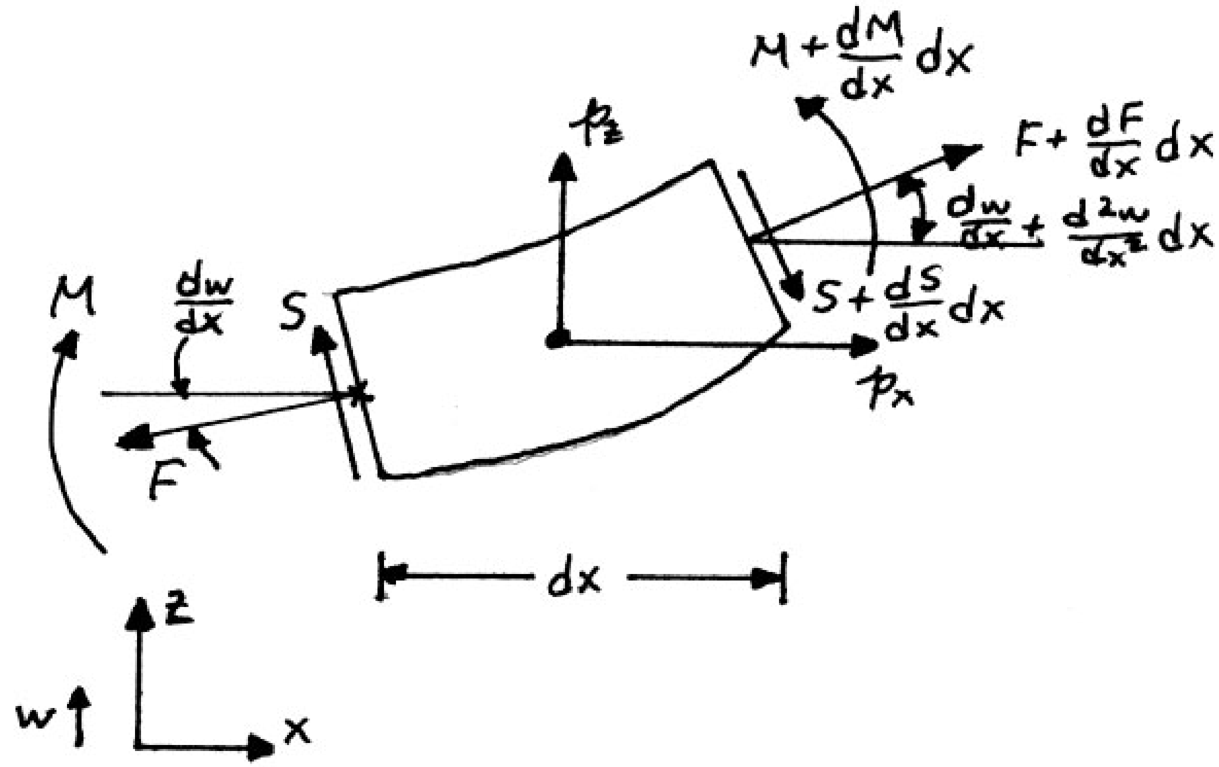


(could also have p_y for bending in y direction)

Consider 2-D case:

Cut out a deformed element dx:

Figure 17.2 **Loads and moment acting on deformed infinitesimal element of beam-column**



Assume small angles such that:

$$\sin\frac{dw}{dx} \approx \frac{dw}{dx}$$

$$\cos\frac{dw}{dx} \approx 1$$

Sum forces and moments:

- $\sum F_x = 0 \quad \xrightarrow{+} \quad :$

$$- F + F + \frac{dF}{dx} dx + p_x \, dx$$

$$- S \frac{dw}{dx} + \left(S + \frac{dS}{dx} dx \right) \left(\frac{dw}{dx} + \frac{d^2 w}{dx^2} dx \right) = 0$$

This leaves:

$$\frac{dF}{dx} dx + p_x \, dx + \left(\frac{dS}{dx} \frac{dw}{dx} + S \frac{d^2 w}{dx^2} \right) dx + \underbrace{H.O.T.}_{(dx)^2} = 0$$

$$\Rightarrow \boxed{\frac{dF}{dx} = - p_x - \underbrace{\frac{d}{dx} \left(S \frac{dw}{dx} \right)}_{\text{new term}}} \qquad (17\text{-}1)$$

- $\sum F_z = 0 \uparrow + :$

$$- F \frac{dw}{dx} + \left(F + \frac{dF}{dx} dx \right) \left(\frac{dw}{dx} + \frac{d^2 w}{dx^2} dx \right)$$

$$+ S - \left(S + \frac{dS}{dx} dx \right) + p_z dx = 0$$

This results in:

$$\boxed{\frac{dS}{dx} = p_z + \underbrace{\frac{d}{dx}\left(F \frac{dw}{dx} \right)}_{\text{new term}}} \qquad (17\text{-}2)$$

- $\sum M_y = 0 \circlearrowleft + :$

$$- M + M + \frac{dM}{dx} dx + p_z dx \frac{dx}{2}$$

$$- p_x dx \frac{dw}{dx} \frac{dx}{2} - \left(S + \frac{dS}{dx} dx \right) dx = 0$$

(using the previous equations) this results in:

$$\frac{dM}{dx} = S \qquad (17\text{-}3)$$

Note: same as before (for Simple Beam Theory)

Recall from beam bending theory:

$$M = EI \frac{d^2 w}{dx^2} \qquad (17\text{-}4)$$

Do some manipulating - place (17-4) into (17-3):

$$S = \frac{d}{dx}\left(EI \frac{d^2 w}{dx^2}\right) \qquad (17\text{-}5)$$

and place this into (17-2) to get:

$$\frac{d^2}{dx^2}\left(EI \frac{d^2 w}{dx^2}\right) - \frac{d}{dx}\left(F \frac{dw}{dx}\right) = p_z \qquad (17\text{-}6)$$

Basic differential equation for Beam-Column --
(Bending equation -- fourth order differential equation)

--> To find the axial force F(x), place (17-5) into (17-1):

$$\frac{dF}{dx} = -p_x - \frac{d}{dx}\left[\frac{dw}{dx}\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]$$

For w small, this latter part is a second order term in w and is therefore negligible

Thus:

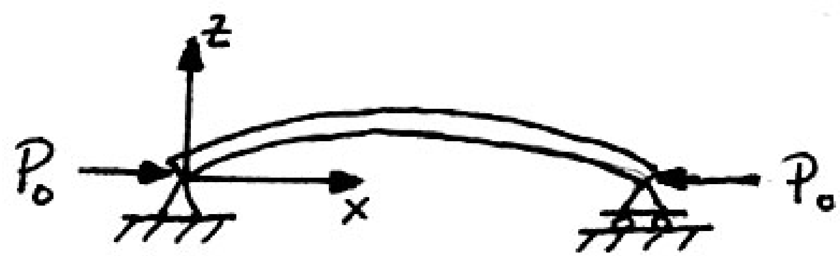
$$\boxed{\frac{dF}{dx} = -p_x} \qquad (17\text{-}7)$$

Note: Solve this equation first to find F(x) distribution and use that in equation (17-6)

Examples of solution to Equation (17-7)

- End compression P_o

Figure 17.3 **Simply-supported column under end compression**



p_x = 0

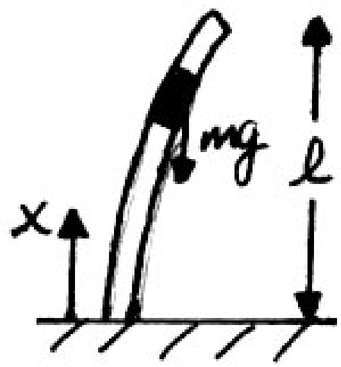
$$\frac{dF}{dx} = 0 \Rightarrow F = C_1$$

find C_1 via boundary condition @x = 0, $F = -P_o = C_1$

$$\Rightarrow F = -P_o$$

- Beam under its own weight

Figure 17.4 **Representation of end-fixed column under its own weight**



$p_x = -mg$

$$\frac{dF}{dx} = +mg \Rightarrow F = mgx + C_1$$

boundary condition: @ $x = \ell$, $F = 0$

So: $mg\ell + C_1 = 0 \Rightarrow C_1 = -mg\ell$

$\Rightarrow F = -mg\,(\ell - x)$

- Helicopter blade

Figure 17.5 **Representation of helicopter blade**

(radial force due to rotation)

similar to previous case

Once have F(x), proceed to solve equation (17-6). Since it is fourth order, need four boundary conditions (two at each end of the beam-column)

--> same possible boundary conditions as previously enumerated

Notes:

- When EI --> 0, equation (17-6) reduces to:

$$-\frac{d}{dx}\left(F\frac{dw}{dx}\right) = p_z$$

this is a **string** (second order $\Rightarrow$ only need two boundary conditions -- one at each end)

(also note that a string cannot be clamped since it cannot carry a moment)

- If F = 0, get:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = p_z$$

and for EI constant:

$$EI\frac{d^4w}{dx^4} = p_z \qquad \text{(basic bending equation)}$$

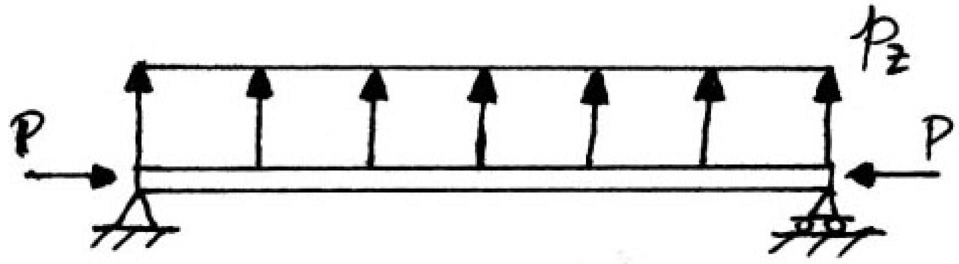
- For p_z = 0, EI constant, and F constant (= -P), get:

$$EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} = 0 \qquad \text{(basic buckling equation)}$$

Buckling of Beam-Column

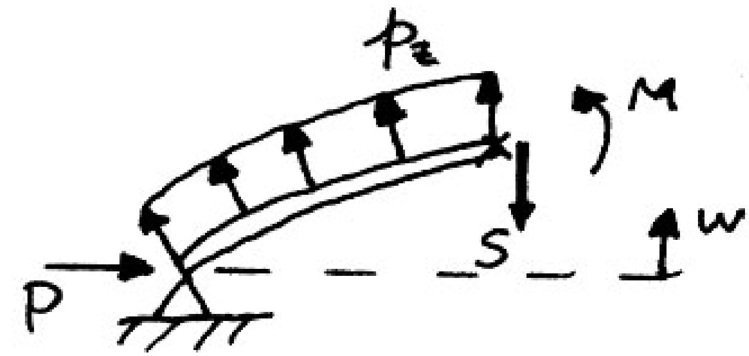
Consider the overall geometry (assume beam-column initially straight)

Figure 17.6 **Representation of general configuration of beam-column**



Cut the beam-column:

Figure 17.7 **Representation of beam-column with cut to determine stress resultants**



$$\sum M = 0 : \quad M - M_{primary} + P\,w = 0$$

$M_{primary}$ → due to transverse loading

Pw → secondary moment (due to deflection) gives:

$$M = EI \frac{d^2w}{dx^2} = M_{primary} - Pw$$

for transverse loading:

$$\frac{d^2}{dx^2}\left(EI \frac{d^2w}{dx^2}\right) - \frac{d}{dx}\left(F \frac{dw}{dx}\right) = p_z$$

integrate twice with F = - P = C_1

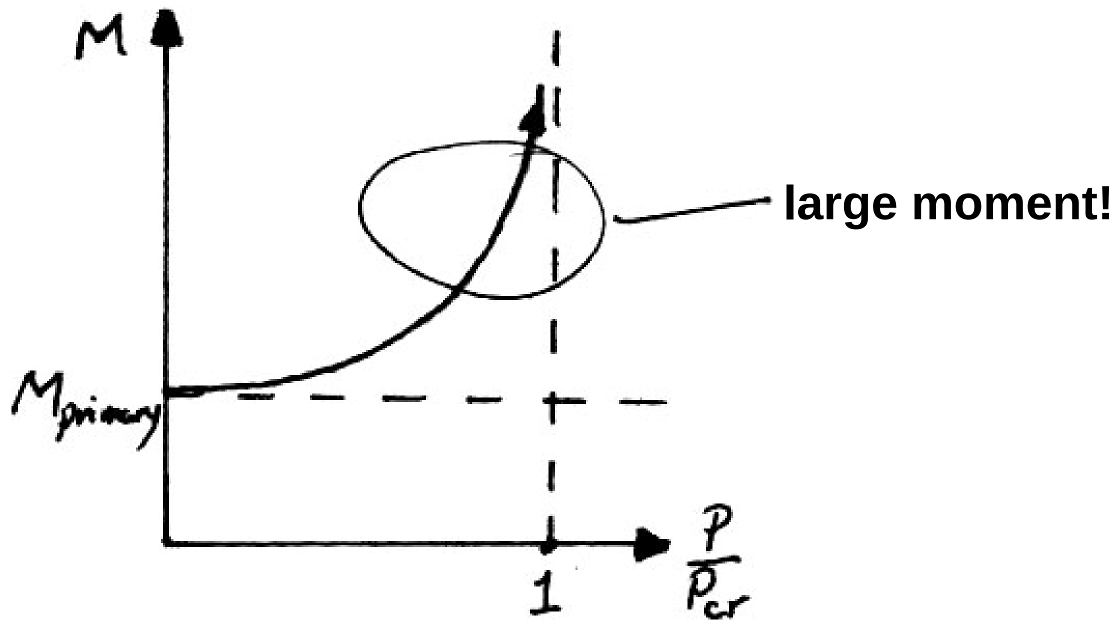
$$EI \frac{d^2w}{dx^2} + Pw = M_{primary}$$

↖ same equation as by doing equilibrium

Solve this by:

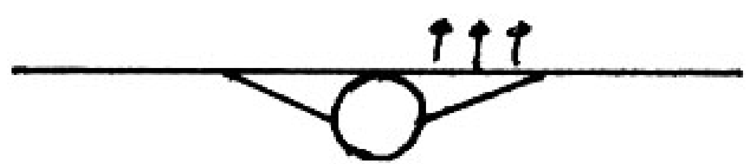
- getting homogenous solution for w
- getting particular solution for $M_{primary}$
- applying boundary condition

Figure 17.8 **Representation of moment(s) versus applied load for beam-column**

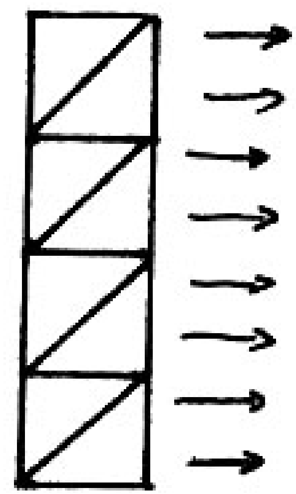


Examples

- "Old" airplanes w/struts



- Space structure undergoing rotation



Final note: The beam-column is an important concept and the moments in a beam-column can be much worse/higher than beam theory or a perfect column alone