# Unit 14

# Behavior of General (*including* Unsymmetric Cross-section) Beams

Readings:

| | |
|---|---|
| Rivello | 7.1 - 7.5, 7.7, 7.8 |
| T & G | 126 |

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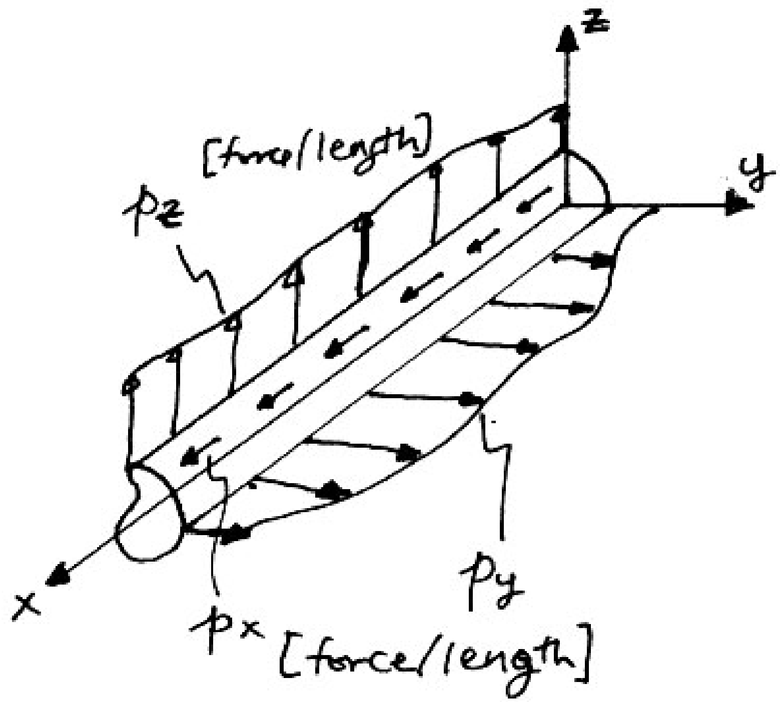
Earlier looked at Simple Beam Theory in which one considers a beam in the x-z plane with the beam along the x-direction and the load in the z-direction:

***Figure 14.1*** **Representation of Simple Beam**


Now look at a more general case:

- Loading can be in any direction
- Can resolve the loading to consider transverse loadings $p_y(x)$ and $p_z(x)$; and axial loading $p_x(x)$
- Include a temperature distribution T(x, y, z)

*Figure 14.2* **Representation of General Beam**




Maintain several of the same definitions for a beam and basic assumptions.

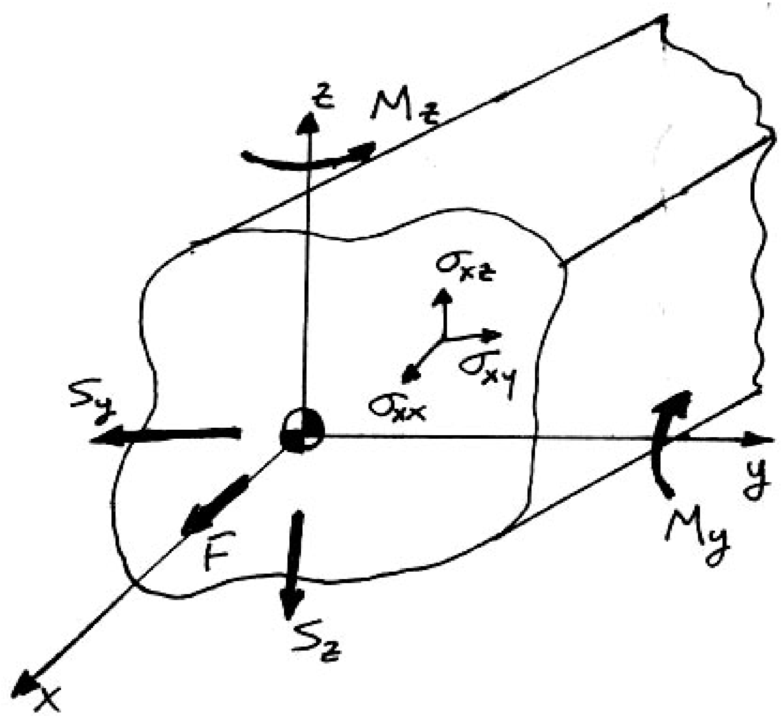
- Geometry: length of beam (x-dimension) greater than y and z dimensions
- Stress State: $\sigma_{xx}$ is the only "important" stress; $\sigma_{xy}$ and $\sigma_{xz}$ found from equilibrium equations, but are secondary in importance
- Deformation: plane sections remain plane and perpendicular to the midplane after deformation (Bernouilli-Euler Hypothesis)

## Definition of stress resultants

Consider a cross-section along x:

***Figure 14.3*** **Representation of cross-section of general beam**

**Place axis @ center of gravity of section**




where:

$$F = \iint \sigma_{xx}\, dA$$

$$S_y = -\iint \sigma_{xy}\, dA$$

$$S_z = -\iint \sigma_{xz}\, dA$$

$$M_y = -\iint \sigma_{xx}\, z\, dA$$

$$M_z = -\iint \sigma_{xx}\, y\, dA$$

These are resultants!

The values of these resultants are found from statics in terms of the loading $p_x$, $p_y$, $p_z$, and applying the boundary conditions of the problem

## Deformation

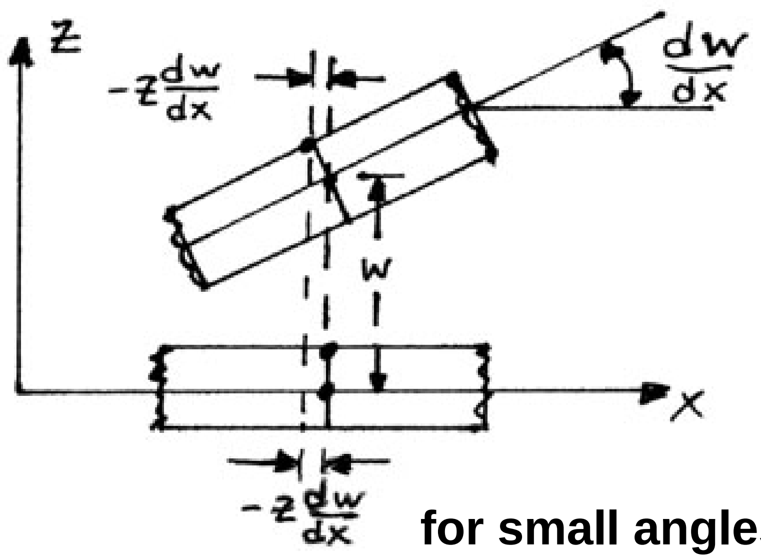
Look at the deformation. In the case of Simple Beam Theory, had:

$$u = -z \frac{dw}{dx}$$

where u is the displacement along the x-axis.

This comes from the picture:

***Figure 14.4*** **Representation of deformation in Simple Beam Theory**

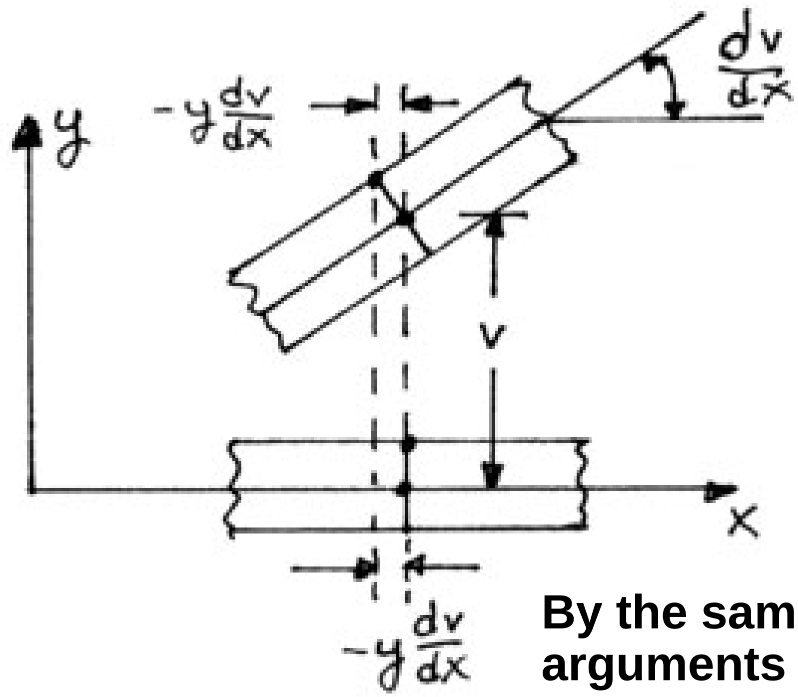



**for small angles**

Now must add two other contributions…..

1. Have the same situation in the x-y plane

***Figure 14.5*** **Representation of bending displacement in x-y plane**

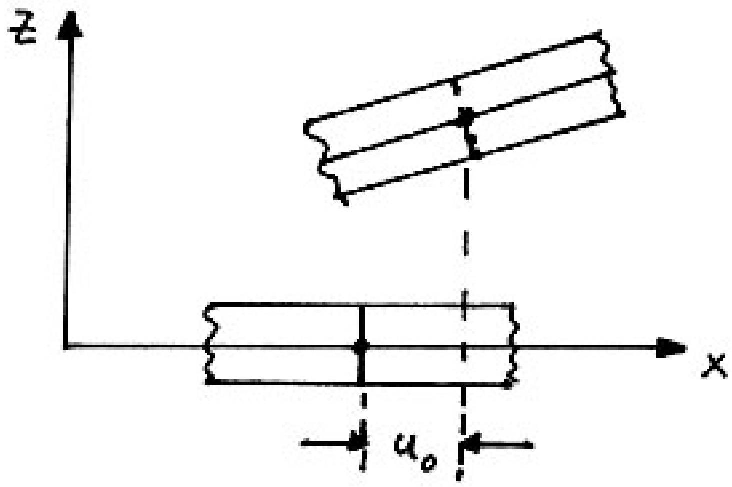



**By the same geometrical arguments used previously**

where v is the displacement in the y-direction

2. Allow axial loads, so have an elongation in the x-direction due to this. Call this $u_0$:

***Figure 14.6*** **Representation of axial elongation in x-z plane**




$u_0$, v, w are the deformations of the midplane

Thus:

$$u(x,y,z) = u_0 - \underbrace{y\frac{dv}{dx}}_{\text{bending about z-axis}} - \underbrace{z\frac{dw}{dx}}_{\text{bending about y-axis}}$$

$$v(x,y,z) = v(x)$$

$$w(x,y,z) = w(x)$$

v and w are constant at any cross-section location, x

## Stress and Strain

From the strain-displacement relation, get:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{d u_0}{dx} + y\left(-\frac{d^2 v}{dx^2}\right) + z\left(-\frac{d^2 w}{dx^2}\right)$$

(these become total derivatives as there is no variation of the displacement in y and z)

for functional ease, write:

$$f_1 = \frac{d u_0}{d x}$$

$$f_2 = -\frac{d^2 v}{d x^2}$$

$$f_3 = -\frac{d^2 w}{d x^2}$$

Caution: Rivello uses $C_1$, $C_2$, $C_3$. These are not constants, so use $f_i \Rightarrow f_i(x)$ (functions of x)

Thus:

$$\varepsilon_{xx} = f_1 + f_2 y + f_3 z$$

Then use this in the stress-strain equation (orthotropic or "lower"):

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} + \alpha \, \Delta T$$

(include temperature effects)

Note: "ignore" thermal strains in y and z. These are of "secondary" importance.

Thus:

$$\sigma_{xx} = E \varepsilon_{xx} - E \alpha \, \Delta T$$

and using the expression for $\varepsilon_x$:

$$\boxed{\sigma_{xx} = E \left( f_1 + f_2 y + f_3 z \right) - E \alpha \, \Delta T}$$

Can place this expression into the expression for the resultants (force and moment) to get:

$$F = \iint \sigma_{xx}\, dA = f_1 \iint E\, dA + f_2 \iint E y\, dA + f_3 \iint E z\, dA - \iint E \alpha \Delta T\, dA$$

$$-M_z = \iint \sigma_{xx}\, y\, dA = f_1 \iint E y\, dA + f_2 \iint E y^2\, dA + f_3 \iint E yz\, dA - \iint E \alpha \Delta T y\, dA$$

$$-M_y = \iint \sigma_{xx}\, z\, dA = f_1 \iint E z\, dA + f_2 \iint E y z\, dA + f_3 \iint E z^2\, dA - \iint E \alpha \Delta T z\, dA$$

(Note: $f_1$, $f_2$, $f_3$ are functions of x and integrals are in dy and dz, so these come outside the integral sign).

Solve these equations to determine $f_1(x)$, $f_2(x)$, $f_3(x)$:

Note: Have kept the modulus, E, within the integral since will allow it to vary across the cross-section

Orthotropic Beams: Same comments as applied to Simple Beam Theory. The main consideration is the longitudinal modulus, so these equations can be applied.

# Modulus-Weighted Section Properties/Areas

Introduce "modulus weighted area":

$$dA^* = \frac{E}{E_1} dA \qquad \text{(vary in y and z)}$$

where:

$A^*$ = modulus weighted area

$E$ = modulus of that area

$E_1$ = some reference value of modulus

Using this in the equations for the resultants, we get:

$$F + \iint E \alpha \Delta T \, dA = E_1 \left\{ f_1 \iint dA^* + f_2 \iint y \, dA^* + f_3 \iint z \, dA^* \right\}$$

$$-M_z + \iint E \alpha \Delta T \, y \, dA = E_1 \left\{ f_1 \iint y \, dA^* + f_2 \iint y^2 \, dA^* + f_3 \iint y z \, dA^* \right\}$$

$$-M_y + \iint E \alpha \Delta T \, z \, dA = E_1 \left\{ f_1 \iint z \, dA^* + f_2 \iint y z \, dA^* + f_3 \iint z^2 \, dA^* \right\}$$

Now define these "modulus-weighted" section properties:

$$\iint dA^* = A^*$$ modulus-weighted area

$$\iint y \, dA^* = \bar{y}^* A^*$$

$$\iint z \, dA^* = \bar{z}^* A^*$$

$$\iint y^2 \, dA^* = I_z^*$$ modulus-weighted moment of inertia about z-axis

$$\iint z^2 \, dA^* = I_y^*$$ modulus-weighted moment of inertia about y-axis

$$\iint y z \, dA^* = I_{yz}^*$$ modulus-weighted <u>product</u> <u>of</u> <u>inertia</u> ("cross" moment of inertia)

Also have "Thermal Forces" and "Thermal Moments". These have the same "units" as forces and moments but are due to thermal effects and can then be treated *analytically* as forces and moments:

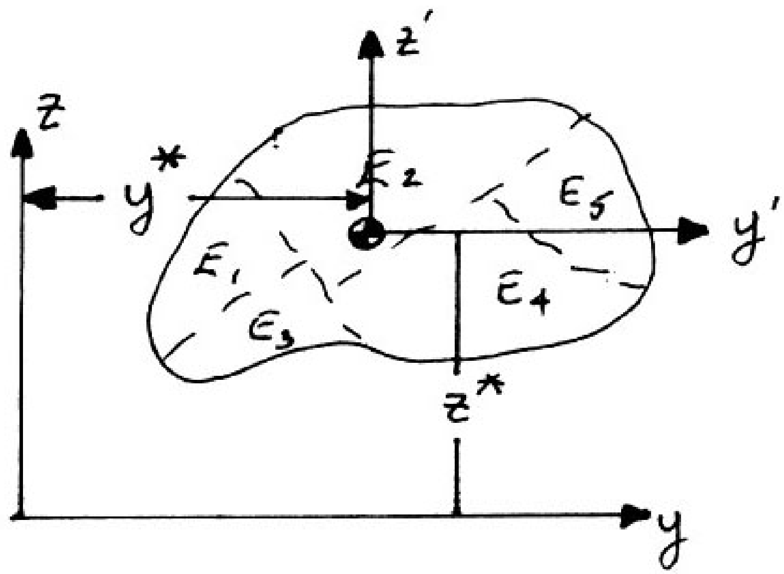
$$F^T = \iint E \alpha \Delta T \, dA$$

$$M_y^T = -\iint E \alpha \Delta T z \, dA \qquad M_z^T = -\iint E \alpha \Delta T y \, dA$$

<u>Note</u>: Cannot use the modulus-weighted section properties since $\alpha$ may also vary in y and z along with E.

In the definition of the section properties, have used a $\bar{y}^*$ and $\bar{z}^*$. These are the location of the "<u>modulus-weighted centroid</u>" referred to some coordinate system

***Figure 14.7*** **Representation of general beam cross-section with pieces with different values of modulus**



The modulus-weighted centroid is defined by:

$$\frac{1}{A^*}\iint y \, dA^* = \bar{y}^*$$

$$\frac{1}{A^*}\iint z \, dA^* = \bar{z}^*$$

These become 0 if one uses the modulus-weighted centroid as the origin

(Note: like finding center of gravity but use E rather than $\rho$)

If one uses the modulus-weighted centroid as the origin, the equations reduce to:

$$\left(F + F^T\right) = F^{TOT} = E_1 f_1 A^*$$

$$-\left(M_z + M_z^T\right) = -M_z^{TOT} = E_1\left(f_2 I_z^* + f_3 I_{yz}^*\right)$$

$$-\left(M_y + M_y^T\right) = -M_y^{TOT} = E_1\left(f_2 I_{yz}^* + f_3 I_y^*\right)$$

(Note: Rivello uses $F^*$, $M_y^*$, $M_z^*$ for $F^{TOT}$, $M_y^{TOT}$, $M_z^{TOT}$)

Recall that:

$$f_1 = \frac{d u_0}{dx}$$

$$f_2 = -\frac{d^2 v}{dx^2}$$

$$f_3 = -\frac{d^2 w}{dx^2}$$

## Motivation for "modulus-weighted' section properties

A beam may not have constant material properties through the section. Two possible ways to vary:

1. Continuous variation

The modulus may be a continuous function of y and z:

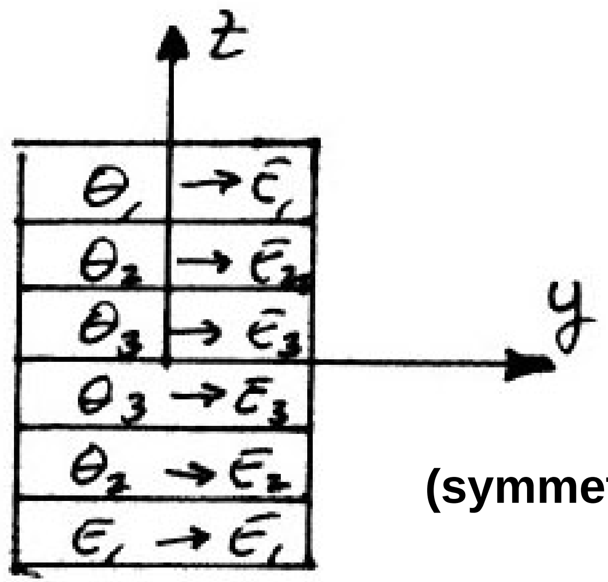
E = E(y, z)

Example: Beam with a large thermal gradient and four different properties through the cross-section

## 2. Stepwise variation

A composite beam which, although it's made of the same material, has different modulus, $E_x$, through-the-thickness as the fiber orientation varies from ply to ply.

***Figure 14.8*** **Representation of cross-section of laminated beam with different modulus values through the thickness**

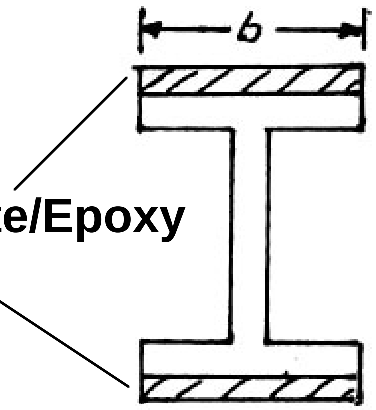



**(symmetric)**

A method of putting material to its best use is called:

# "Selective Reinforcement"

***Figure 14.9*** **Representation of selective reinforcement of an I-beam**




**Unidirectional Graphite/Epoxy cap reinforcements**

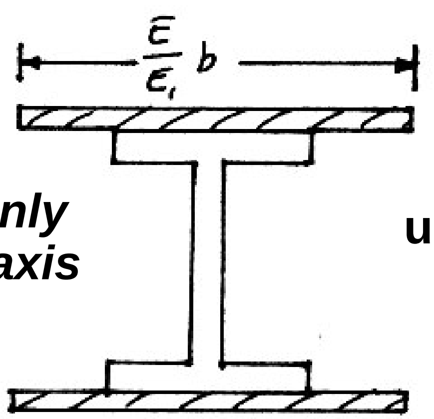
**(E = 20 Msi)**

**Aluminum I-beam (E = 10 Msi)**

**Furthest from neutral axis ⇒ best resistance to bending**

Using aluminum as the reference, analyze as follows

***Figure 14.10*** **Representative cross-section with aluminum as base**




***Representation good only in direction parallel to axis about which I is taken.***

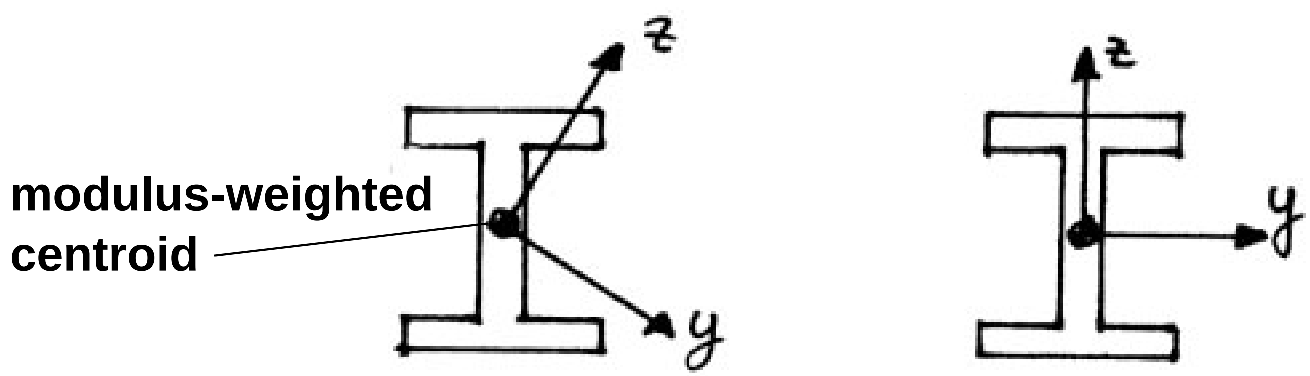
**use $E_1$ to analyze**

$$\frac{E}{E_1} b = \frac{20\ msi}{10\ msi} b = 2b$$

Principal Axes of structural cross-section:

There is a set of y, z axes such that the product of inertia $\left(I_{yz}^*\right)$ is zero. These are the principal axes (section has axes of symmetry)

***Figure 14.11*** **Representation of principal axes of structural cross-section**




$I_{yz}^* \neq 0$ $\qquad$ $I_{yz}^* = 0$

(use Mohr's circle transformation)

If analysis is conducted in the principal axes, the equations reduce to:

$$f_1 = \frac{F^{TOT}}{E_1 A^*} = \frac{d u_0}{dx}$$

$$f_2 = -\frac{M_z^{TOT}}{E_1 I_z^*} = -\frac{d^2 v}{dx^2}$$

$$f_3 = -\frac{M_y^{TOT}}{E_1 I_y^*} = -\frac{d^2 w}{dx^2}$$

These equations can be integrated to find the deflections $u_0$, v and w

These expressions for the $f_i$ can be placed into the equation for $\sigma_{xx}$ to obtain:

$$\sigma_{xx} = \frac{E}{E_1}\left\{ \frac{F^{TOT}}{A^*} - \frac{M_z^{TOT}}{I_z^*} y - \frac{M_y^{TOT}}{I_y^*} z - E_1 \alpha \, \Delta T \right\}$$

where y,z are principal axes for the section

If the axes are <u>not</u> principal axes ($I_{yz}^* \neq 0$), have:

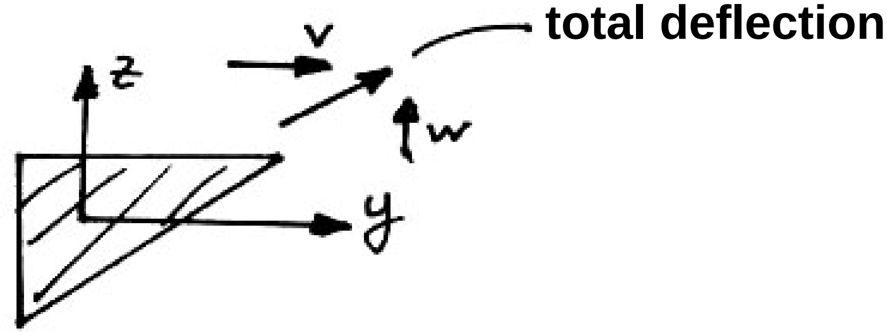
$$f_1 = \frac{F^{TOT}}{E_1 A^*} = \frac{d u_0}{dx} \qquad \text{(no change)}$$

$$f_2 = \frac{-I_y^* M_z^{TOT} + I_{yz}^* M_y^{TOT}}{E_1 \left( I_y^* I_z^* - I_{yz}^{*\,2} \right)} = -\frac{d^2 v}{dx^2}$$

$$f_3 = \frac{-I_z^* M_y^{TOT} + I_{yz}^* M_z^{TOT}}{E_1 \left( I_y^* I_z^* - I_{yz}^{*\,2} \right)} = -\frac{d^2 w}{dx^2}$$

<u>Note</u>: If $I_{yz}^* \neq 0$ then both w and v are present for $M_y$ or $M_z$ only

***Figure 14.12*** **Representation of deflection of cross-section not in principal axes**



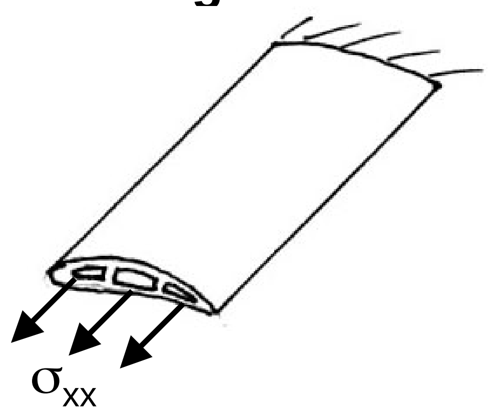
In this case, the expression for the stress is rather long:

$$\sigma_{xx} = \frac{E}{E_1}\left\{ \frac{F^{TOT}}{A^*} - \frac{\left[ I_y^* M_z^{TOT} - I_{yz}^* M_y^{TOT} \right] y}{I_y^* I_z^* - I_{yz}^{*\,2}} - \frac{\left[ I_z^* M_y^{TOT} - I_{yz}^* M_z^{TOT} \right] z}{I_y^* I_z^* - I_{yz}^{*\,2}} - E_1 \alpha \Delta T \right\}$$

"Engineering Beam Theory"
(Non-Principal Axes)

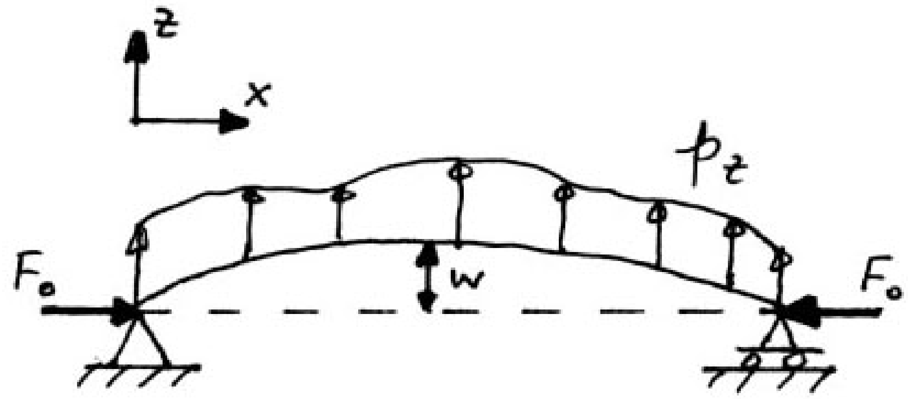
Analysis is good for high aspect ratio structure (e.g. a wing)

***Figure 14.13*** **Representation of wing as beam**



Note: this analysis neglects the effect of the axial Force F on the Bending Moment. This became important as the deflection w (or v) becomes large:

***Figure 14.14*** **Representation of large deflection when axial force and bending deflection couple**




**$F_0$ = axial force**

$$M = \underbrace{M_{due\ to\ p_z}}_{\text{Primary Bending Moment}} - \underbrace{wF_0}_{\text{Secondary Moment}}$$

Secondary moment known as "membrane effect". Can particularly become important if $F_o$ is near buckling load (will talk about when talk about beam-column)

## Shear Stresses

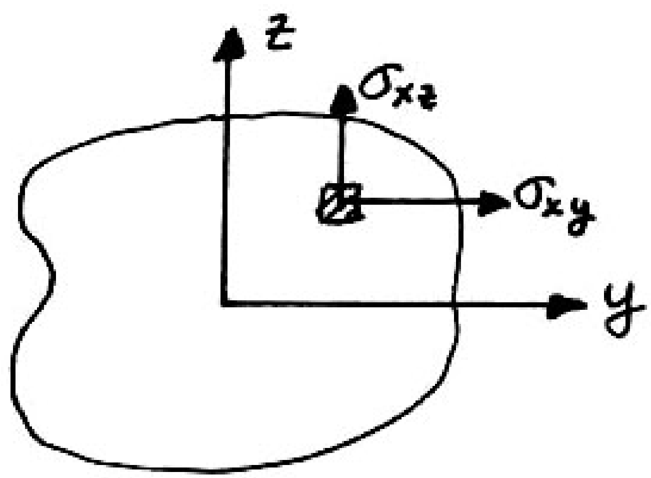
The shear stresses ($\sigma_{xy}$ and $\sigma_{xz}$) can be obtained from the equilibrium equations:

$$\frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = -\frac{\partial \sigma_{xx}}{\partial x}$$

$$\frac{\partial \sigma_{xy}}{\partial x} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} = 0$$

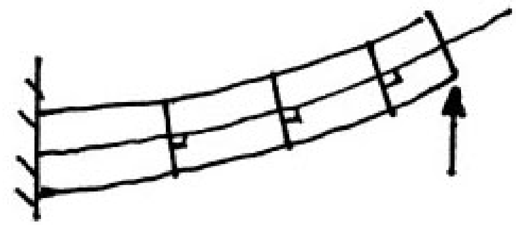
***Figure 14.15*** **Representation of cross-section of general beam**



These shear stresses (called “transverse” shear stresses) cause “small” additional shearing contributions to deflections

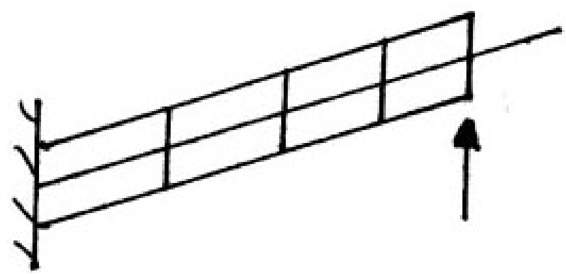
***Figure 14.16*** **Representation of pure bending and pure shearing of a beam**

**Pure Bending -->**



Plane sections remain plane and perpendicular to midplane

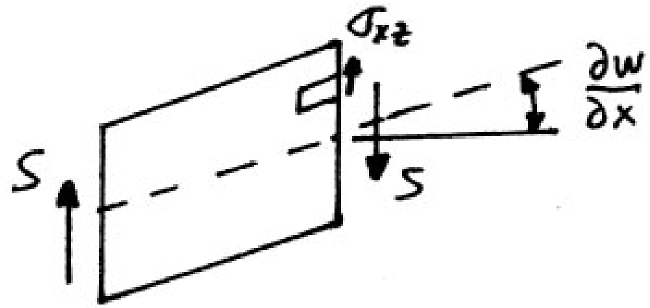
**Pure Shearing -->**



Plane sections remain plane but <u>not</u> perpendicular to midplane

Consider a beam section under “pure shearing”…

***Figure 14.17*** **Representation of deformation of beam cross-section under pure shearing**



$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\sigma_{xz}}{G}$$

↑ engineering shear strain

Average $\frac{\partial w}{\partial x}$ over cross-section:

$$\left(\frac{\partial w}{\partial x}\right)_{ave} = \frac{\iint \left(\frac{\partial w}{\partial x}\right) dA}{\iint dA} = \frac{\frac{1}{G} \overbrace{\iint \sigma_{xz}\, dA}^{S_z}}{A} = -\frac{S_z}{G A}$$

Actually, from energy considerations, one should average:

$$\left(\frac{\partial w}{\partial x}\right)_{ave} = \frac{\iint \left(\frac{\partial w}{\partial x}\right)^2 dA}{\iint \frac{\partial w}{\partial x}\, dA} \approx -\frac{S_z}{G A_e}$$

↑ "effective area"

For a Rectangular Cross-Section: $A_e \approx 0.83\ A$

Then, "pure shearing" deflections, $w_s$, governed by:

$$\frac{d w_s}{d x} = - \frac{S}{G A_e}$$

$$w_s = - \int_0^x \frac{S}{G A_e} + C_1$$

↑ evaluated from boundary conditions

The total beam deflection is the sum of the two contributions:

$W_T = W_B + W_S$

$$\boxed{w_T = w_B + w_s}$$

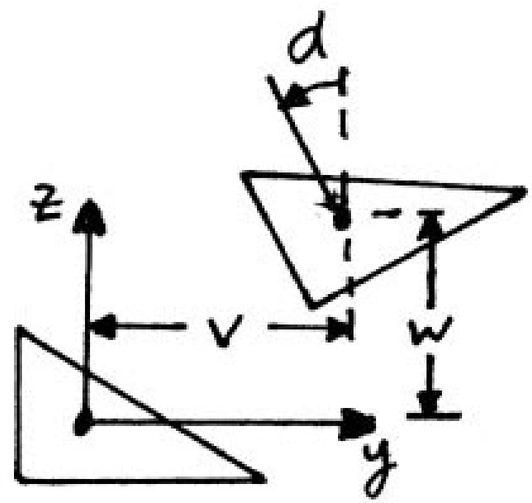
- $w_T$: **total**
- $w_B$: **bending deflection from** $E I \frac{d^2 w_B}{dx^2} = M$
- $w_s$: **shearing deflection from** $G A_e \frac{d w_S}{d x} = - S$

Ordinarily, $w_s$ is small for ordinary rectangular beams (and can be ignored). But, for thin-walled sections, $w_s$ can become important

(worse for composites since $G_{xz} << E_x$)

In addition to "bending" and "shearing", the section may also twist through an angle $\alpha$

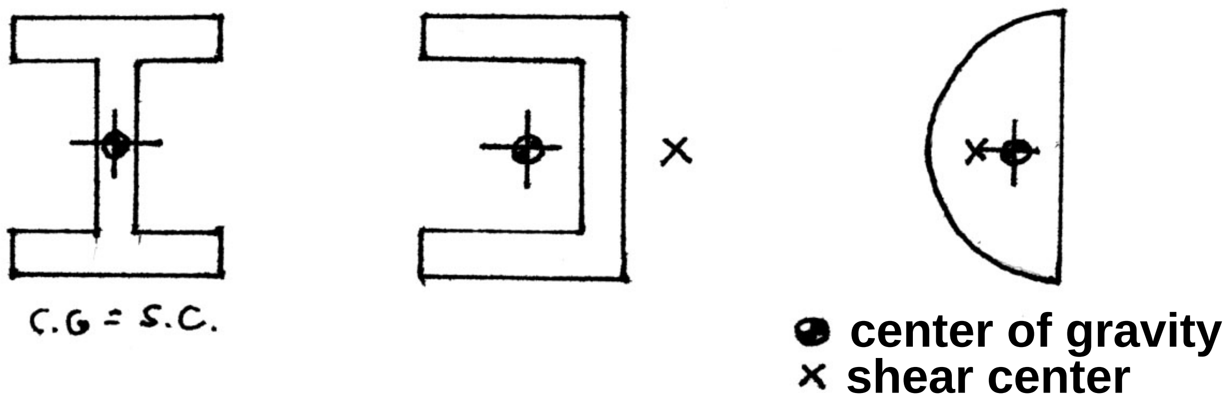
***Figure 14.18*** **Representation of twisting of beam cross-section**




However, there exists a **<u>Shear</u>** **<u>Center</u>** for every section. If the load is applied at the shear center, the section translates but does not twist.

(<u>Note</u>: shear center not necessarily center of gravity or centroid)

***Figure 14.19*** **Representation of some beam cross-sections with various locations of center of gravity and shear center**



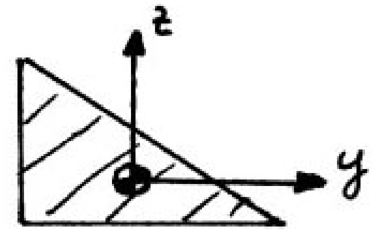

If this condition is not met, then generally bending and twisting will *couple*. But there is a class of cross-sections (thin-walled) where bending and shearing/torsion can be decoupled. Will pursue this next.

Wrap-up discussion by considering examples of common cross-sections with principal axes aligned such that $I_{yz} = 0$ *(see Handout #4a)*

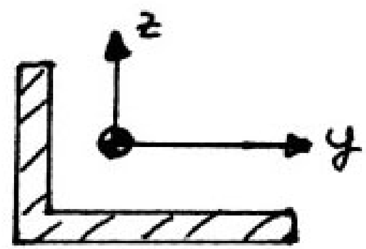
These are in contrast to common cross-sections not principal axes ($I_{yz} \neq 0$)

## *Figure 14.19* Some cross-sections generally not in principal axes

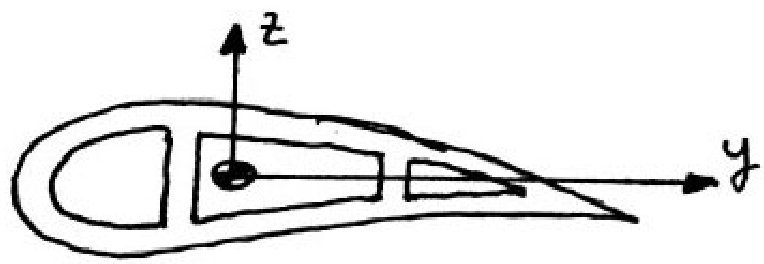
**Triangle**




**Angle**




**Wing Section**



# --> Finally

What are the limitations to the Engineering Beam Theory as developed?

- Shear deflections small (can get first order cut at this)
- No twisting (load along shear center) -- otherwise torsion and bending couple
- Deflections small
  - o – No moment due to axial load ($P_w$)
  - o – Angles small such that $\sin\phi \approx \phi$
  - --> will consider next order effect when discuss buckling and postbuckling
  - --> consideration will stiffen (membrane effect) structure
- Did not consider $\varepsilon_{zz}$ (Poisson's effect)