# Unit 13
# Review of Simple Beam Theory

<u>Readings</u>:

Review Unified Engineering notes on Beam Theory

| | |
|---|---|
| BMP | 3.8, 3.9, 3.10 |
| T & G | 120-125 |

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# IV. General Beam Theory

We have thus far looked at:

- in-plane loads
- torsional loads

In addition, structures can carry loads by *bending*. The 2-D case is a *plate*, the simple 1-D case is a *beam*. Let's first review what you learned in Unified as **Simple Beam Theory**

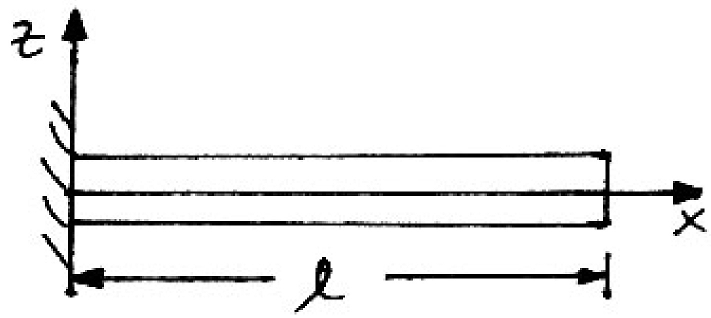
## (review of) Simple Beam Theory

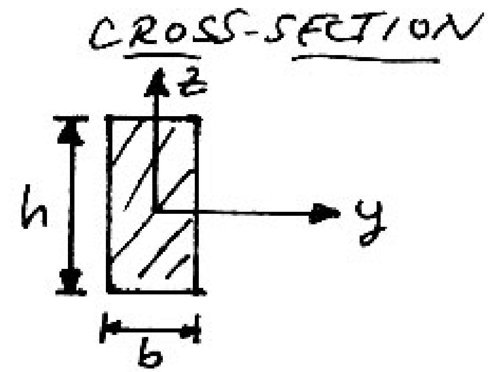
A *beam* is a bar capable of carrying loads in bending. The loads are applied transverse to its longest dimension.

Assumptions:

1. Geometry

## *Figure 13.1* General Geometry of a Beam




a) long & thin ⇒ $\ell >> b, h$

b) loading is in z-direction

c) loading passes through "shear center" ⇒ no torsion/twist
(we'll define this term later and relax this constraint.)

d) cross-section can vary along x

2. Stress state

a) $\sigma_{yy}, \sigma_{yz}, \sigma_{xy} = 0$ ⇒ no stress in y-direction

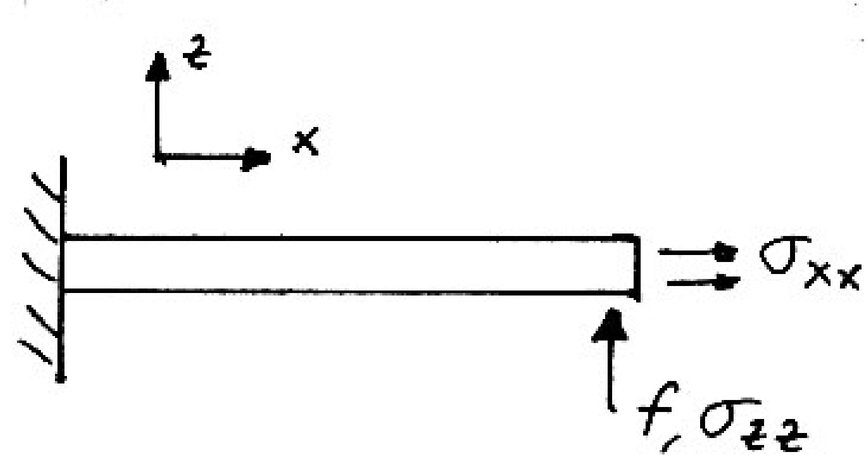
b) $\sigma_{xx} >> \sigma_{zz}$
• $\sigma_{xz} >> \sigma_{zz}$ ⇒ only significant stresses are $\sigma_{xx}$ and $\sigma_{xz}$

Note: there is a load in the z-direction to cause these stresses, but generated $\sigma_{xx}$ is much larger (similar to pressurized cylinder example)

Why is this valid?

Look at moment arms:

***Figure 13.2*** **Representation of force applied in beam**




$\sigma_{xx}$ moment arm is order of (h)

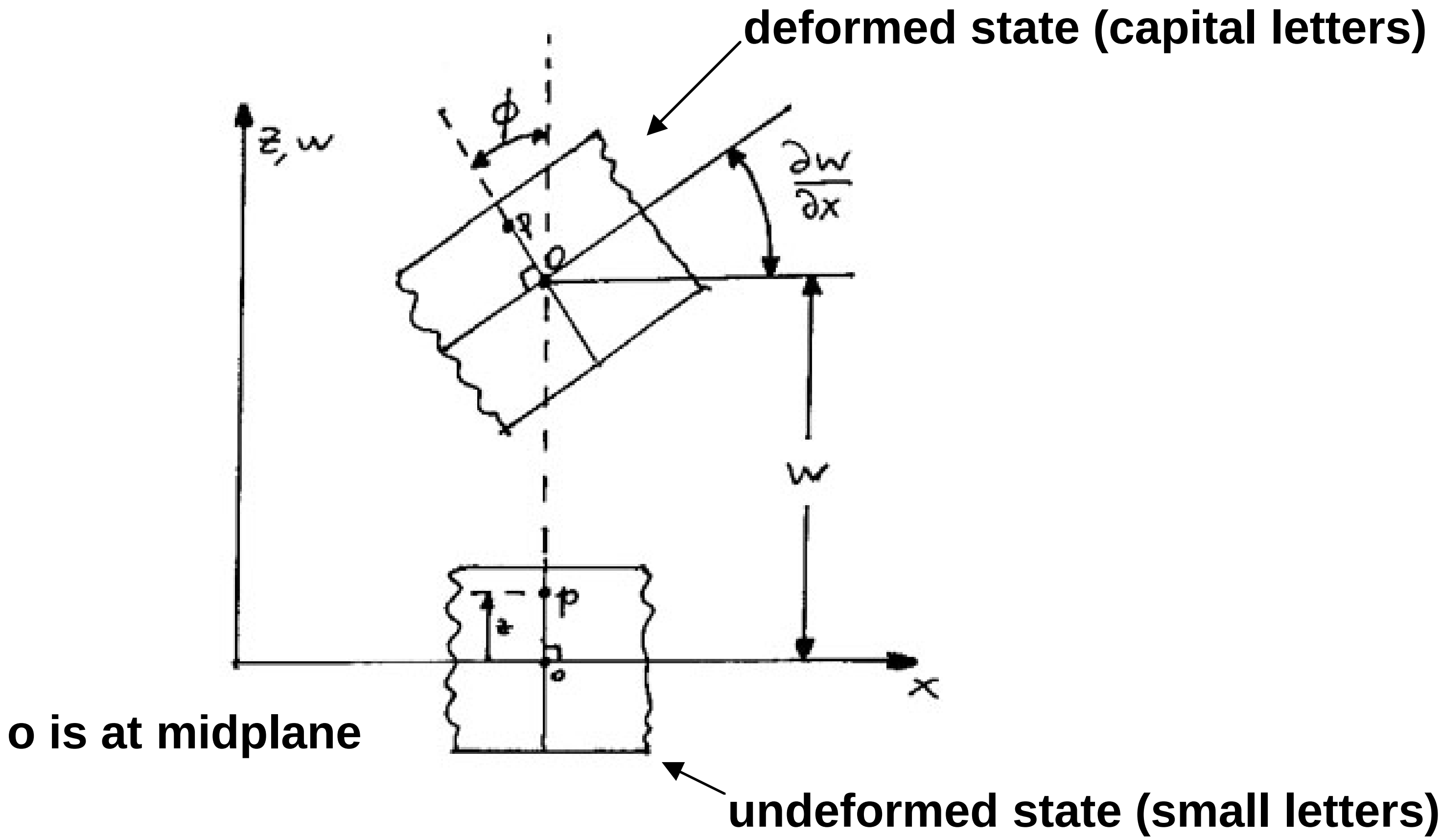
$\sigma_{zz}$ moment arm is order of ($\ell$)

and $\ell >> h$

$\Rightarrow \sigma_{xx} >> \sigma_{zz}$ for equilibrium

## 3. Deformation

***Figure 13.3*** **Representation of deformation of cross-section of a beam**




define: w = deflection of midplane (function of x only)

a) Assume plane sections remain plane and perpendicular to the midplane after deformation

"Bernouilli - Euler Hypothesis" ~ 1750

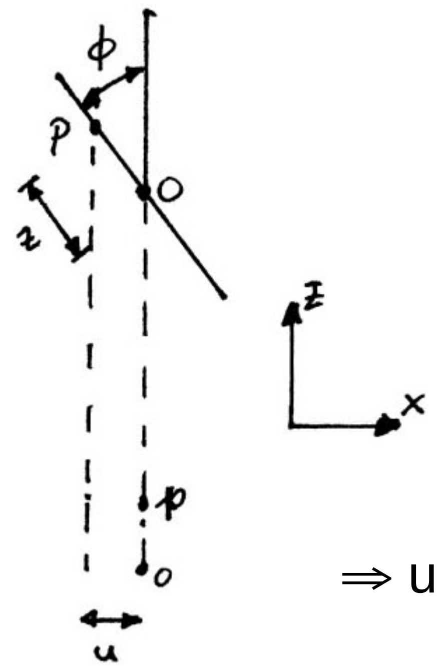
b) For small angles, this implies the following for deflections:

$$u(x,y,z) \approx -z\phi \approx -z\frac{dw}{dx} \qquad (13 - 1)$$

$$\left(\phi = \frac{dw}{dx}\right)$$

total derivative since it does not vary with y or z

*Figure 13.4* **Representation of movement in x-direction of two points on same plane in beam**




$\Rightarrow$ u = -z sin $\phi$ Note direction of u relative to +x direction and for $\phi$ small:

$\Rightarrow$ u = -z $\phi$

$$v(x,y,z) = 0$$

$$w(x,y,z) \approx w(x) \qquad (13 - 2)$$

Now look at the strain-displacement equations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{d^2 w}{dx^2} \qquad (13 - 3)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$ (no deformation through thickness)

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$$

$$\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} = 0$$

Now consider the stress-strain equations (for the time being consider isotropic…extend this to orthotropic later)

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \qquad (13 - 4)$$

$$\varepsilon_{yy} = -\frac{\nu \sigma_{xx}}{E}$$ <-- small inconsistency with previous

$$\varepsilon_{zz} = -\frac{\nu \sigma_{xx}}{E}$$ <-- small inconsistency with previous

$$\varepsilon_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy} = 0$$

$$\varepsilon_{yz} = \frac{2(1+\nu)}{E}\sigma_{yz} = 0$$

$$\varepsilon_{zx} = \frac{2(1+\nu)}{E}\sigma_{zx}$$ <-- inconsistency again!

We get around these inconsistencies by saying that $\varepsilon_{yy}$, $\varepsilon_{zz}$, $\varepsilon_{xz}$ are <u>very</u> small but not quite zero. This is an ***approximation***. We will evaluate these later on.

## 4. Equilibrium Equations

Assumptions:

a) no body forces

b) equilibrium in y-direction is "ignored"

c) x, z equilibrium are satisfied in an average sense

So:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \qquad (13 - 5)$$

$$0 = 0 \qquad \text{(y -equilibrium)}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \qquad (13 - 6)$$

Note, average equilibrium equations:

$$\iint_{face} \left[ \quad \text{Eq. } (13 - 6) \quad \right] dy\, dz \Rightarrow \frac{dS}{dx} = p \qquad (13 - 6a)$$

$$\iint_{face} z \left[ \quad \text{Eq. } (13 - 5) \quad \right] dy\, dz \Rightarrow \frac{dM}{dx} = S \qquad (13 - 5a)$$

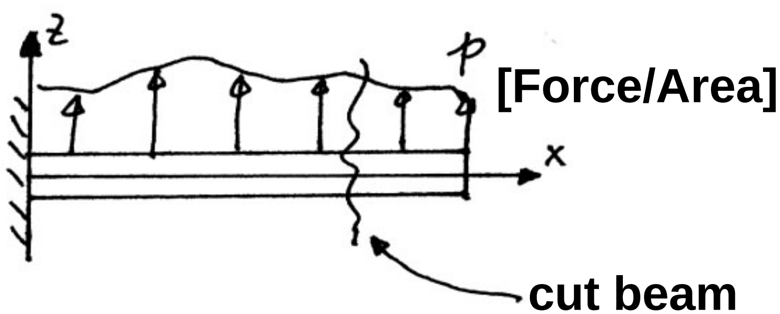
These are the Moment, Shear, Loading relations where the stress resultants are:

Axial Force $$F = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} b \, dz \qquad (13 - 7)$$

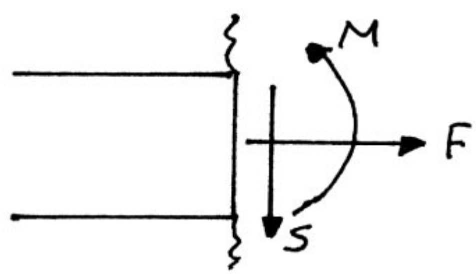
Shear Force $$S = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} b \, dz \qquad (13 - 8)$$

Bending Moment $$M = -\int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_{xx} b \, dz \qquad (13 - 9)$$

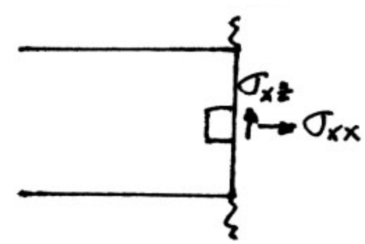
*Figure 13.8* **Representation of Moment, Shear and Loading on a beam**




**(F, S, M found from statics)**




**and**



So the *final important equations of Simple Beam Theory* are:

$$\varepsilon_{xx} = -z \frac{d^2 w}{dx^2} = \frac{\partial u}{\partial x} \qquad (13 - 3)$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \qquad (13 - 4)$$

$$\frac{dS}{dx} = p \qquad (13 - 6a)$$

$$\frac{dM}{dx} = S \qquad (13 - 5a)$$

--> How do these change if the material is orthotropic?

We have assumed that the properties along x dominate and have ignored $\varepsilon_{yy}$, etc.

Thus, use $E_L$ in the above equations.

**But**, approximation may not be as good since
$\varepsilon_{yy}$, $\varepsilon_{zz}$, $\varepsilon_{xz}$ may be large and really not close enough to zero to be assumed approximately equal to zero

## Solution of Equations

using (13 - 3) and (13 - 4) we get:

$$\sigma_{xx} = E\,\varepsilon_{xx} = -Ez\frac{d^2w}{dx^2} \qquad (13 - 10)$$

Now use this in the expression for the axial force of equation (13 - 7):

$$F = -E\frac{d^2w}{dx^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} z\,b\,dz$$

$$= -E\frac{d^2w}{dx^2}\left.\frac{z^2}{2}b\right]_{-\frac{h}{2}}^{\frac{h}{2}} = 0$$

***No axial force in beam theory***

(Note: something that carries axial and bending forces is known as a *beam-column*)

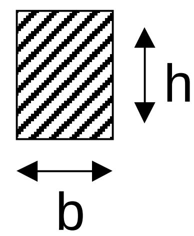
Now place the stress expression (13 - 10) into the moment equation (13 - 9):

$$M = E\frac{d^2w}{dx^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2\,b\,dz$$

definition: $$I = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 \, b \, dz$$ moment of inertia of cross-section

for rectangular cross-section:

$$I = \frac{b h^3}{12}$$ [length$^4$]




Thus:

$$M = E I \frac{d^2 w}{dx^2} \qquad (13 - 11)$$

"Moment - Curvature Relation"

--> Now place equation (13 - 11) into equation (13 - 10) to arrive at:

$$\sigma_{xx} = - E z \frac{M}{E I}$$

$$\Rightarrow \sigma_{xx} = - \frac{M z}{I} \qquad (13 - 12)$$

Finally, we can get an expression for the shear stress by using equation (13 - 5):

$$\frac{\partial \sigma_{xz}}{\partial z} = - \frac{\partial \sigma_{xx}}{\partial x} \qquad (13 - 5)$$

Multiply this by b and integrate from z to h/2 to get:

$$\int_z^{\frac{h}{2}} b \frac{\partial \sigma_{xz}}{\partial z} dz = -\int_z^{\frac{h}{2}} \frac{\partial \sigma_{xx}}{\partial x} b\, dz$$

$$\Rightarrow b\left[\underbrace{\sigma_{xz}(h/2)}_{=0} - \sigma_{xz}(z)\right] = -\underbrace{\int_z^{\frac{h}{2}} \frac{\partial}{\partial x}\left(-\frac{Mz}{I}\right)}_{= -\frac{z}{I}\frac{dM}{dx} = S} b\, dz$$

(from boundary condition of no stress on top surface)

(13 - 13)

This all gives:

$$\Rightarrow \boxed{\sigma_{xz} = -\frac{SQ}{Ib}}$$

where:

$$Q = \int_z^{\frac{h}{2}} z\, b\, dz \qquad = \text{Moment of the area above the center}$$

↖ function of z - - maximum occurs at z = 0

Summarizing:

$$\frac{dS}{dx} = p$$

$$\frac{dM}{dx} = S$$

$$\sigma_{xx} = -\frac{Mz}{I}$$

$$\sigma_{xz} = -\frac{SQ}{Ib}$$

$$M = EI\frac{d^2w}{dx^2}$$

Notes:

- $\sigma_{xx}$ is linear through thickness and zero at midpoint
- $\sigma_{xz}$ has parabolic distribution through thickness with maximum at midpoint
- Usually $\sigma_{xx} >> \sigma_{zz}$

Solution Procedure

1. Draw free body diagram
2. Calculate reactions
3. Obtain shear via (13 - 6a) and then $\sigma_{xz}$ via (13 - 13)
4. Obtain moment via (13 - 5a) and then $\sigma_{xx}$ via (13 - 12) and deflection via (13 - 11)

**NOTE:** steps 2 through 4 must be solved simultaneously if loading is indeterminate

Notes:

- Same formulation for orthotropic material except
  - Use $E_L$
  - Assumptions on $\varepsilon_{\alpha\beta}$ may get worse
- Can also be solved via stress function approach

- For beams with discontinuities, can solve in each section separately and join (match boundary conditions)

***Figure 13.6*** **Example of solution approach for beam with discontinuity**


$$E I \frac{d^2 w_A}{dx^2} = M_A \qquad E I \frac{d^2 w_B}{dx^2} = M_B$$

$$E I w_A = \ldots + C_1 x + C_2 \qquad E I w_B = \ldots + C_3 x + C_4$$

--> Subject to Boundary Conditions:

@ $x = 0$, $w = w_A = 0$

@ $x = x_1$, $\left\{ \begin{array}{c} w_A = w_B \\ \dfrac{dw_A}{dx} = \dfrac{dw_B}{dx} \end{array} \right\}$ displacements and slopes <u>match</u>

@ $x = \ell$, $w = w_B = 0$