Unit 12
Torsion of (Thin) Closed Sections

Readings:

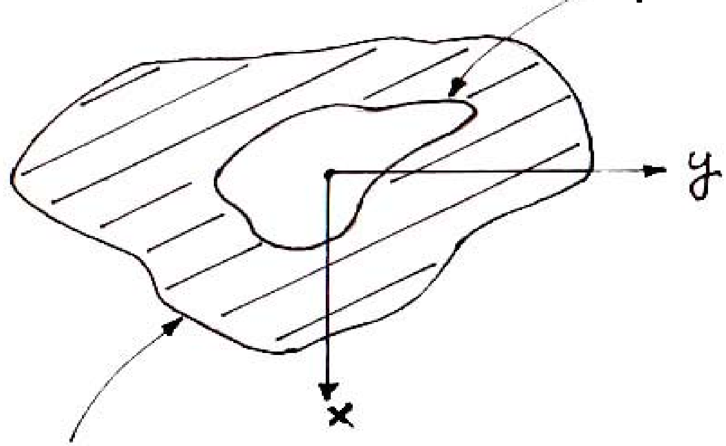
Megson	8.5
Rivello	8.7 (only single cell material),
	8.8 (Review)
T & G	115, 116

Paul A. Lagace, Ph.D.
Professor of Aeronautics & Astronautics
and Engineering Systems

Before we look specifically at thin-walled sections, let us consider the general case (i.e., thick-walled).

Hollow, thick-walled sections:

Figure 12.1 **Representation of a general thick-walled cross-section**



This has more than one boundary (multiply-connected)

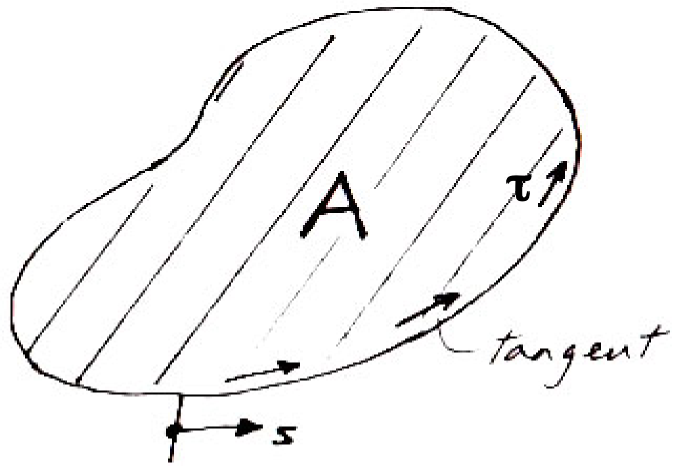
- $d\phi = 0$ on each boundary
- However, $\phi = C_1$ on one boundary and C_2 on the other (they cannot be the same constants for a general solution [there's no reason they should be])

=> *Must somehow be able to relate C_1 to C_2*

It can be shown that around any *closed* boundary:

$$\oint \tau ds = 2AGk \qquad (12\text{-}1)$$

Figure 12.2 **Representation of general closed area**



where:

τ = resultant shear stress at boundary

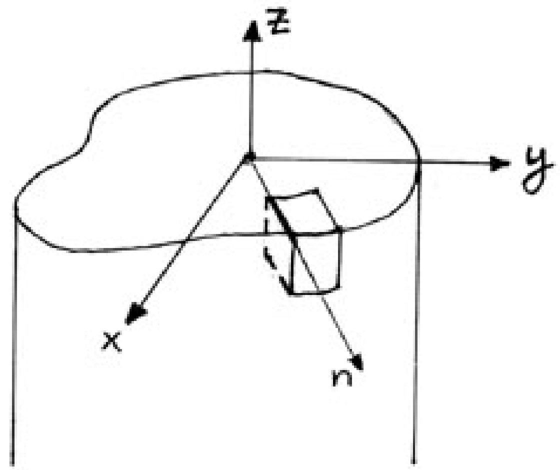
A = Area inside boundary

k = twist rate = $\frac{d\alpha}{dz}$

Notes:

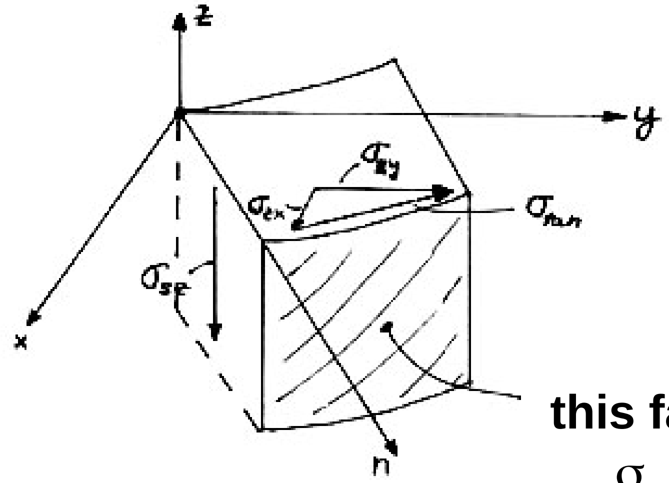
1. The resultant shear stresses at the boundary must be in the direction of the tangents to the boundary
2. The surface traction at the boundary is zero (stress free), but the resultant shear stress is not

Figure 12.3 **Representation of a 3-D element cut with one face at the surface of the body**



To prove Equation (12 - 1), begin by considering a small 3-D element from the previous figure

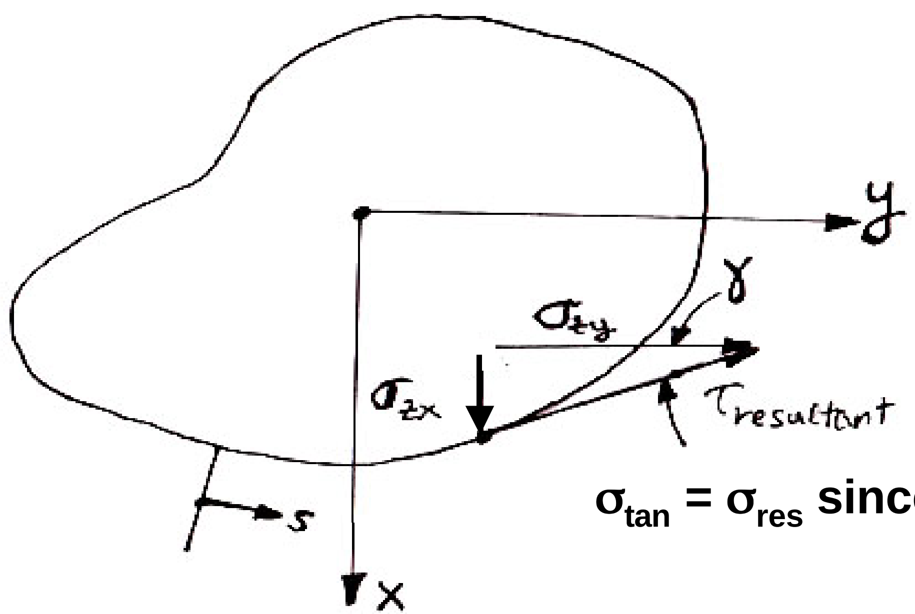
Figure 12.4 **Exploded view of cut-out 3-D elements**



this face is stress free, thus
$\sigma_{normal} = 0$

Look at a 2-D cross-section in the x-y plane:

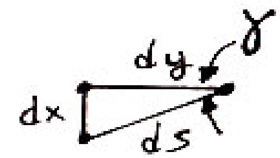
Figure 12.5 **Stress field at boundary of cross-section**



$\sigma_{tan} = \sigma_{res}$ since $\sigma_{normal} = 0$

$$\tau_{\text{resultant}} = \sigma_{zy} \cos\gamma + \sigma_{zx} \sin\gamma$$

geometrically: $\cos\gamma = \dfrac{dy}{ds}$



$$\sin\gamma = \frac{dx}{ds}$$

Thus:

$$\oint \tau \, ds = \oint \left\{ \sigma_{zy} \frac{dy}{ds} ds + \sigma_{zx} \frac{dx}{ds} ds \right\}$$

$$= \oint \sigma_{zy} \, dy + \sigma_{zx} \, dx$$

We know that:

$$\sigma_{zy} = G \left(kx + \frac{\partial w}{\partial y} \right)$$

$$\sigma_{zx} = G \left(-ky + \frac{\partial w}{\partial x} \right)$$

$$\Rightarrow \oint \tau \, ds = \oint G \left(kx + \frac{\partial w}{\partial y} \right) dy + \oint G \left(-ky + \frac{\partial w}{\partial x} \right) dx$$

$$= G \oint \underbrace{\left\{ \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy \right\}}_{= dw} + Gk \oint \left\{ x dy - y dx \right\}$$

We further know that:

$$\oint dw = w \Big] = 0 \quad \text{around closed contour}$$

So we're left with:

$$\oint \tau ds = Gk \oint \left\{ x dy - y dx \right\}$$

Use Stoke's Theorem for the right-hand side integral:

$$\oint \left\{ M dx + N dy \right\} = \iint \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy$$

In this case we have

$$M = -y \Rightarrow \frac{\partial M}{\partial y} = -1$$

$$N = x \Rightarrow \frac{\partial N}{\partial x} = 1$$

We thus get:

$$Gk \oint \left\{ x dy - y dx \right\} = Gk \iint \left[1 - (-1) \right] dx dy$$

$$= Gk \iint 2 dx dy$$

We furthermore know that the double integral of dxdy is the planar area:

$$\iint dxdy = \text{Area} = A$$

Putting all this together brings us back to Equation (12 - 1):

$$\boxed{\oint \tau ds = 2AGk}$$

Q.E.D.

Hence, in the general case we use equation (12 - 1) to relate C_1 and C_2. This is rather complicated and we will not do the general case here. For further information

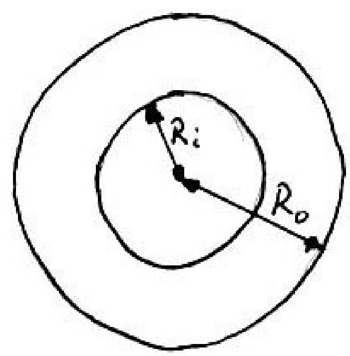
(See Timoshenko, Sec. 115)

We can however consider and do the…

Special Case of a Circular Tube

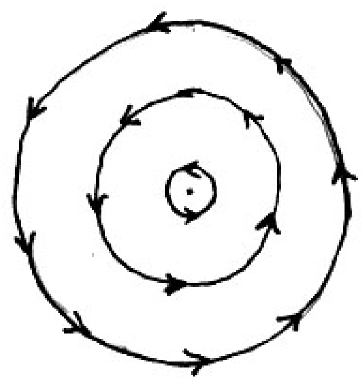
Consider the case of a circular tube with inner diameter R_i and outer diameter R_o

Figure 12.6 **Representation of cross-section of circular tube**



For a solid section, the stress distribution is thus:

Figure 12.7 **Representation of stress "flow" in circular tube**



τ_{res} is directed along circles

The resultant shear stress, τ_{res}, is always tangent to the boundaries of the cross-section

So, we can “cut out” a circular piece (around same origin) without violating the boundary conditions *(of τ_{res} acting tangent to the boundaries)*

Using the solution for a solid section, we subtract the torsional stiffness of the “removed piece” (radius of R_i) from that for the solid section (radius of R_o)

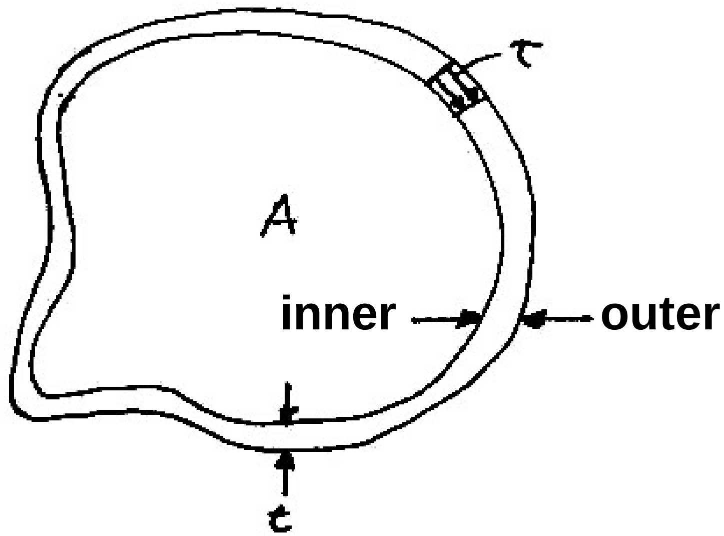
$$J = \frac{\pi R_o^4}{2} - \frac{\pi R_i^4}{2}$$

Exact solution for thick-walled circular tube

let us now consider:

Thin-Walled Closed Sections

Figure 12.7 **Representation of cross-section of thin-walled closed section**



Here, the inner and outer boundaries are nearly parallel ⇒ resultant shear stresses throughout wall are tangent to the median line.

Basic assumption for thin, closed section:

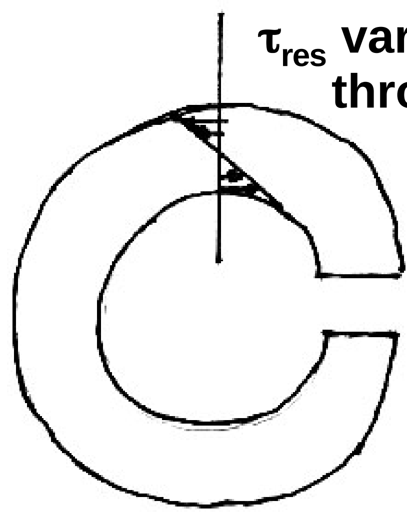
$\tau_{resultant}$ is approximately constant through the thickness t.

For such cases: $A_{outer} \approx A_{inner} \approx A$

Hence: $$\oint_{outer} \tau \, ds \approx \oint_{inner} \tau \, ds \approx 2GkA$$

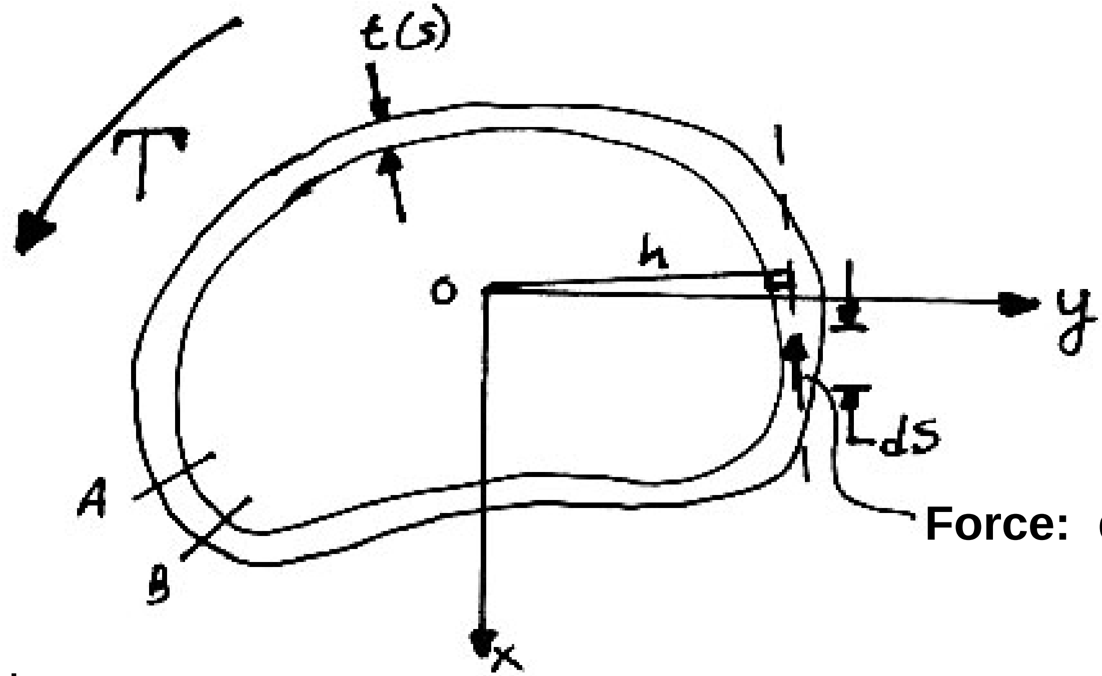
<u>Note</u>: basic difference from singly-connected boundaries (open sections).

Figure 12.9 **Representation of stress distribution through thickness in open cross-section under torsion**



Now, we need to find the boundary conditions:

Figure 12.10 **Representation of forces on thin closed cross-section under torsion**



contribution to torque:

$$dT = h\tau t\, ds$$

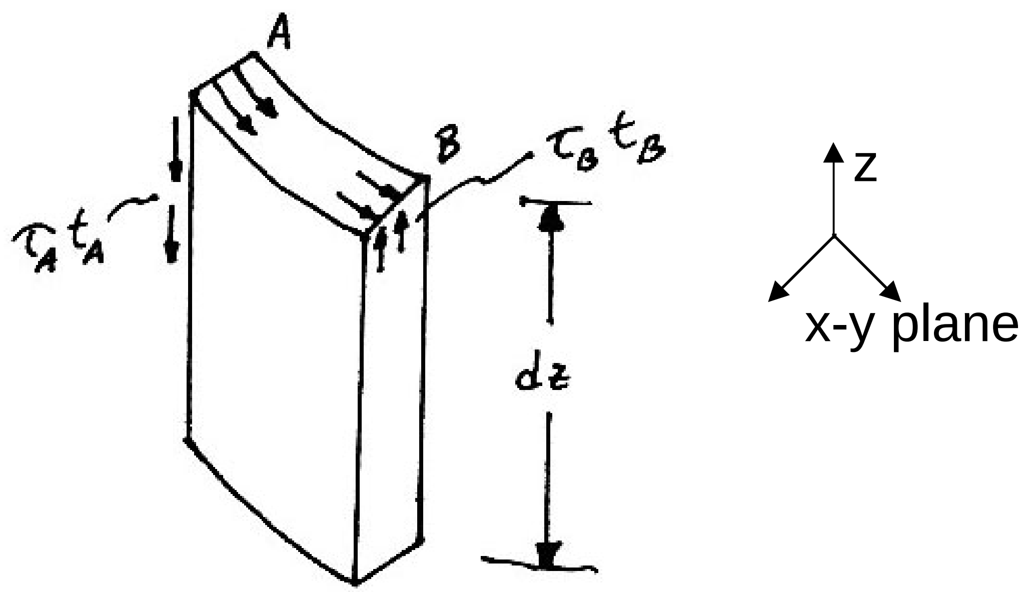
(h = moment arm)

Note: h, τ, t vary with s (around section)

Total torque $= \oint dT = \oint \tau t h\, ds$

<u>But</u> τ t is constant around the section. This can be seen by cutting out a piece of the wall AB.

Figure 12.11 **Representation of infinitesimal piece of wall of thin closed section under torsion**



Use $\sum F_z = 0$ to give:

$$-\tau_A t_A dz + \tau_B t_B dz = 0$$

$$\Rightarrow \tau_A t_A = \tau_B t_B$$

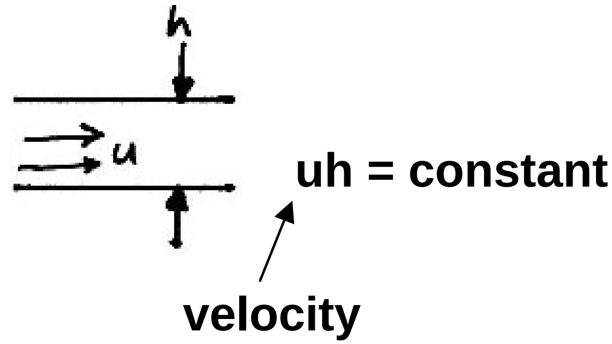
in general: τ t = constant

Define:

$$\boxed{\text{"shear flow"} = q = \tau t} = \text{constant}$$

(we will use the concept of "shear flow" when we deal with shell beams)

Analogy: single 1-D pipe flow



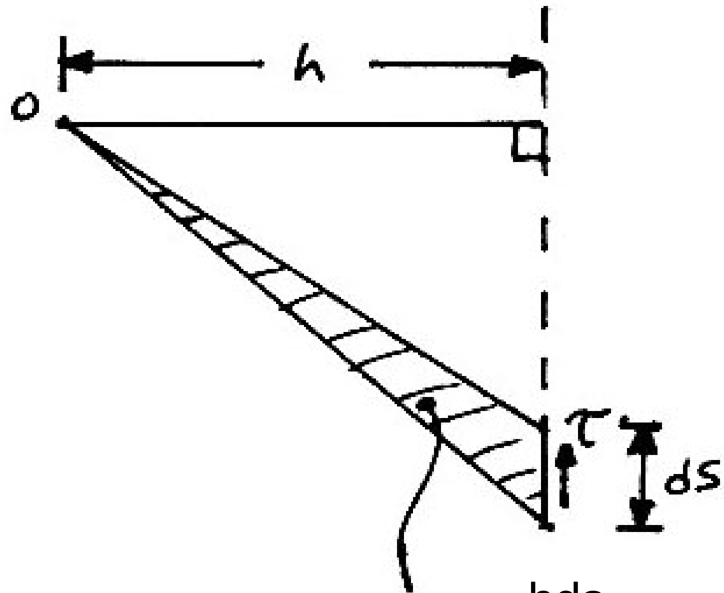
Returning to

$$\oint dT = \oint \tau t h ds$$

since τt = constant gives:

$$\oint dT = \tau t \oint h ds$$

But, hds = 2dA via geometric argument:



$$dA = \frac{hds}{2}$$

$$\left(\frac{\text{height x base}}{2}\right) = \text{Area of Triangle}$$

Finally:

$$\Rightarrow \oint dT = \tau t \oint 2\, dA$$

$$\Rightarrow T = 2 \tau t A$$

$$\Rightarrow \boxed{\tau_{\text{resultant}} = \frac{T}{2At}} \qquad (12 - 2)$$

← Bredt's formula

Now to find the angle of twist, place (12 - 2) into (12 - 1):

$$\oint \frac{T}{2At} ds = Gk2A$$

$$\Rightarrow k = \frac{T}{4A^2G} \oint \frac{ds}{t}$$

This can be rewritten in the standard form:

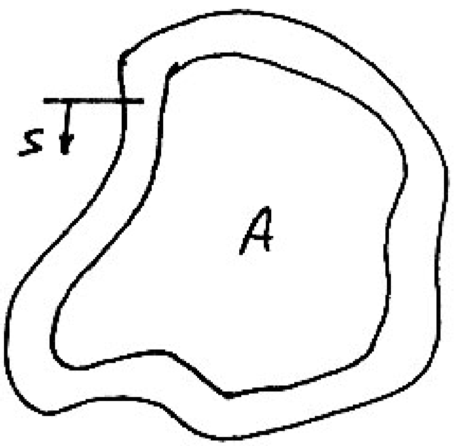
$$k = \frac{d\alpha}{dz} = \frac{T}{GJ}$$

$$\Rightarrow \boxed{J = \frac{4A^2}{\oint \frac{ds}{t}}}$$

(Note: use midline for calculation)

valid for any shape…..

Figure 12.12 Representation of general thin closed cross-section



How good is this approximation?

It will depend on the ratio of the thickness to the overall dimensions of the cross-section (a radius to the center of torsion)

Can explore this by considering the case of a circular case since we have an exact solution:

$$J = \frac{\pi R_o^4 - \pi R_i^4}{2}$$

versus approximation:

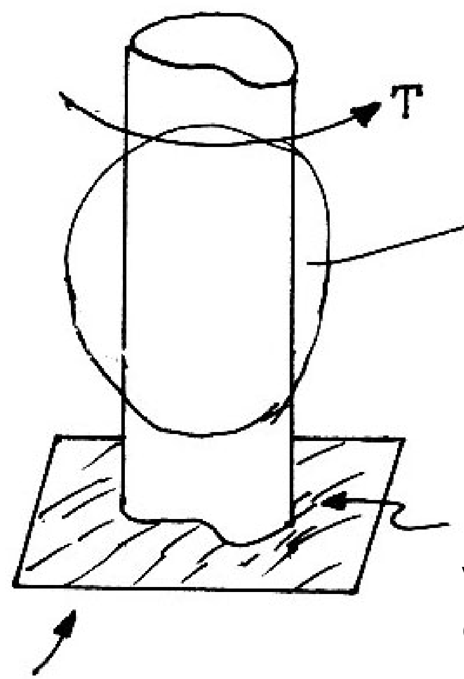
$$J \approx \frac{4A^2}{\oint \frac{ds}{t}}$$

(will explore in home assignment)

Final note on St. Venant Torsion:

When we look at the end constraint (e.g., rod attached at boundary):

Figure 12.13 **Overall view of rod under torsion**



At the base, w = 0. This is a violation of the "free to warp" assumption. Thus, σ_{zz} will be present.

$\Rightarrow$ resort to complex variables

(See Timoshenko & Rivello)