Unit 11
Membrane Analogy (for Torsion)

Readings:

Rivello	8.3, 8.6
T & G	107, 108, 109, 110, 112, 113, 114

Paul A. Lagace, Ph.D.
Professor of Aeronautics & Astronautics
and Engineering Systems

For a number of cross-sections, we cannot find stress functions. However, we can resort to an analogy introduced by Prandtl (1903).

Consider a membrane under pressure p_i

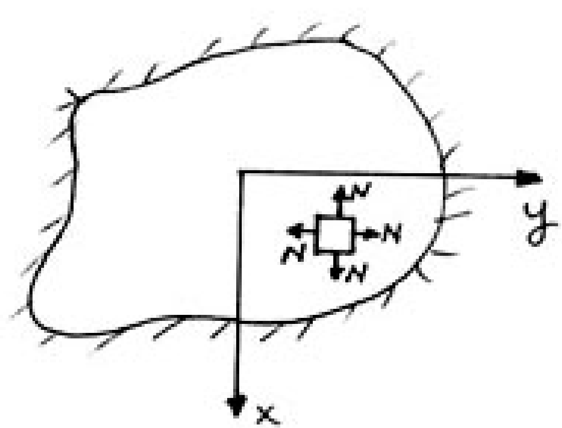
"Membrane": structure whose thickness is small compared to surface dimensions and it (thus) has negligible bending rigidity (e.g. soap bubble)

⇒ membrane carries load via a constant tensile force along itself.

N.B. Membrane is 2-D analogy of a string
(plate is 2-D analogy of a beam)

Stretch the membrane over a cutout of the cross-sectional shape in the x-y plane:

Figure 11.1 **Top view of membrane under pressure over cutout**

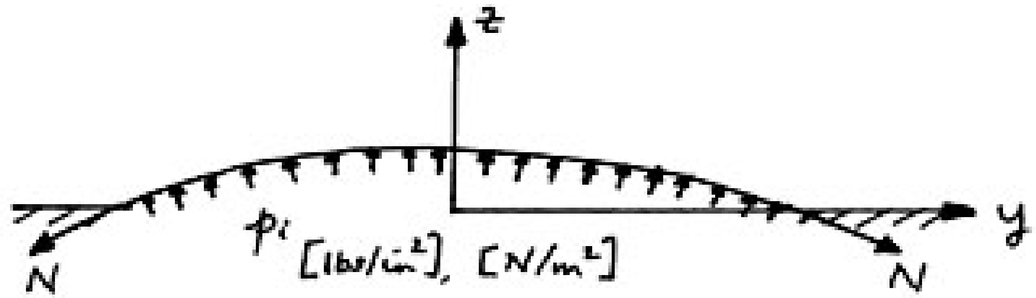


membrane covering a cu

N = constant tension force per unit length [lbs/in] [N/M]

Look at this from the side:

Figure 11.2 **Side view of membrane under pressure over cutout**

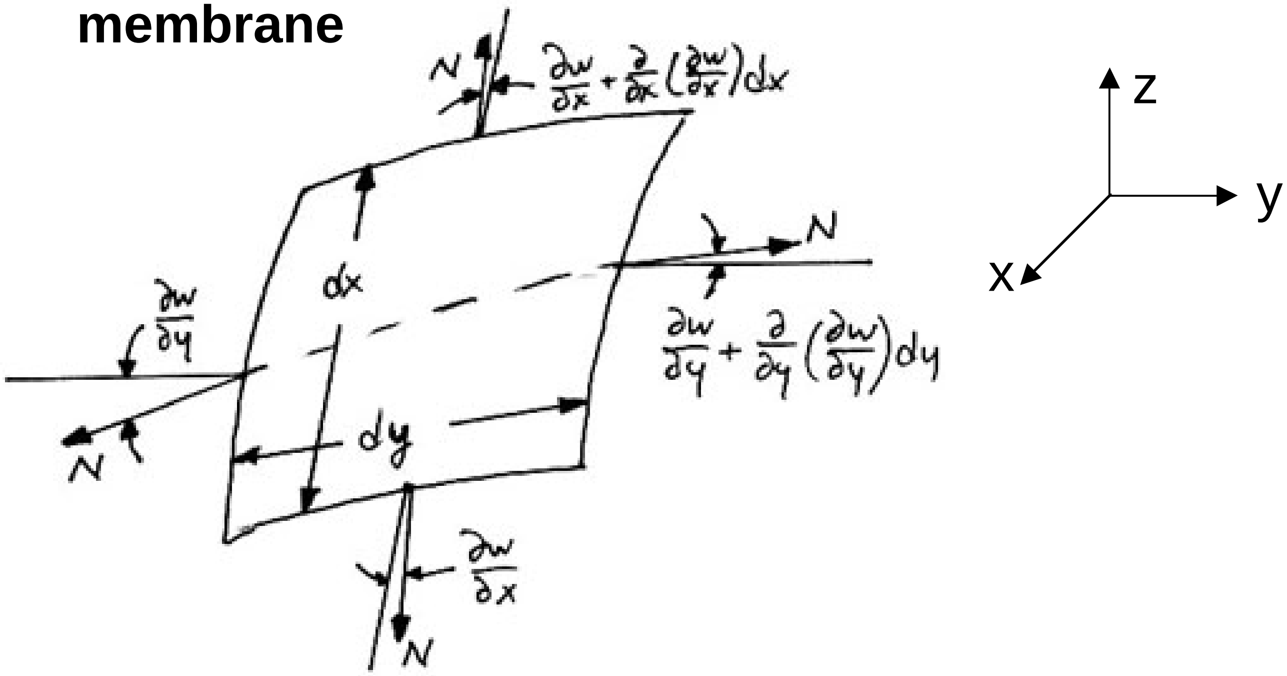


Assume: lateral displacements (w) are small such that no appreciable changes in N occur.

We want to take equilibrium of a small element:

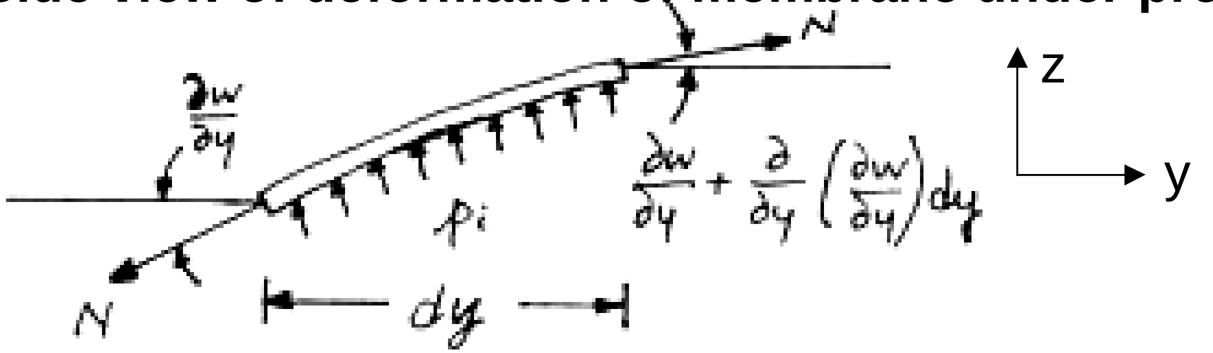
(assume small angles $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$)

Figure 11.3 **Representation of deformation of infinitesimal element of membrane**



Look at side view (one side):

Figure 11.4 **Side view of deformation of membrane under pressure**

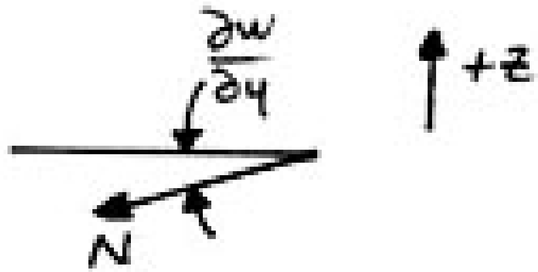


<u>Note</u>: we have similar picture in the x-z plane

We look at equilibrium in the z direction.

Take the z-components of N:

e. g.



z-component = $-N \sin \frac{\partial w}{\partial y}$

↑ note +z direction

for small angle:

$$\sin \frac{\partial w}{\partial y} \approx \frac{\partial w}{\partial y}$$

$\Rightarrow$ z-component $= -N \frac{\partial w}{\partial y}$ (acts over dx face)

With this established, we get:

$$\uparrow + \quad \sum F_z = 0 \Rightarrow \quad p_i \, dxdy \; - \; N \frac{\partial w}{\partial y} dx \; + \; N \left[\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2} dy \right] dx$$

$$- \, N \frac{\partial w}{\partial x} dy \; + \; N \left[\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx \right] dy \; = \; 0$$

Eliminating like terms and canceling out dxdy gives:

$$p_i \; + \; N \frac{\partial^2 w}{\partial y^2} \; + \; N \frac{\partial^2 w}{\partial x^2} \; = \; 0$$

$$\Rightarrow \boxed{\frac{\partial^2 w}{\partial x^2} \; + \; \frac{\partial^2 w}{\partial y^2} \; = \; - \frac{p_i}{N}}$$

Governing Partial Differential Equation for deflection, w, of a membrane

<u>Boundary</u> <u>Condition</u>: membrane is attached at boundary, so w = 0 along contour

$\Rightarrow$ Exactly the same as torsion problem:

	Torsion	Membrane
Partial Differential Equation	$\nabla^2 \phi = 2Gk$	$\nabla^2 w = - p_i / N$
Boundary Condition	$\phi = 0$ on contour	$w = 0$ on contour

Analogy:

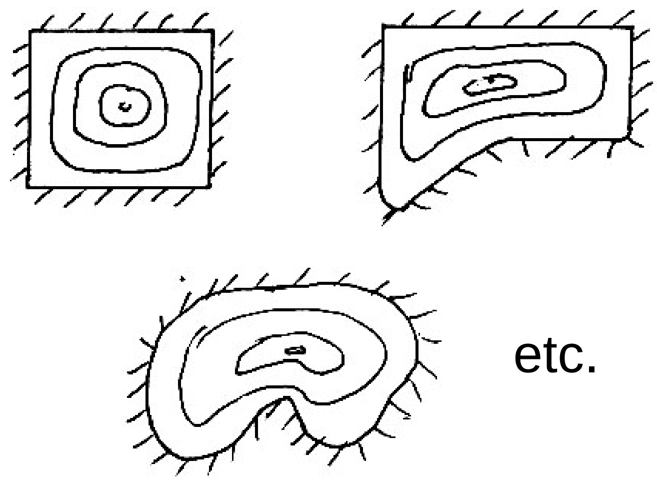
Membrane		Torsion
w	$\rightarrow$	ϕ
p_i	$\rightarrow$	$- k$
N	$\rightarrow$	$\frac{1}{2G}$
$\frac{\partial w}{\partial x}$	$\rightarrow$	$\frac{\partial \phi}{\partial x} = \sigma_{zy}$
$\frac{\partial w}{\partial y}$	$\rightarrow$	$\frac{\partial \phi}{\partial y} = -\sigma_{zx}$
Volume $= \iint w\,dx\,dy$	$\rightarrow$	$-\frac{T}{2}$

<u>Note</u>: for orthotropic, would need a membrane to give different N's in different directions in proportion to G_{xz} and G_{yz}

$\Rightarrow$ Membrane analogy only applies to isotropic materials

- This analogy gives a good "physical" picture for ϕ
- Easy to visualize deflections of membrane for odd shapes

Figure 11.5 **Representation of ϕ and thus deformations for various closed cross-sections under torsion**



Can use (and people have used) elaborate soap film equipment and measuring devices

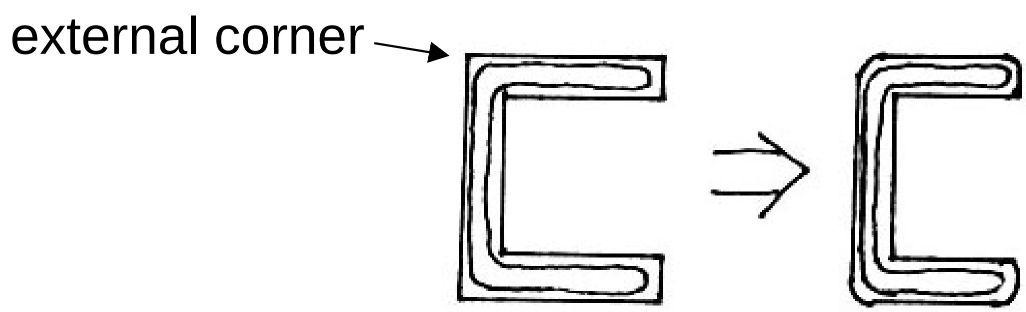
(See Timoshenko, Ch. 11)

From this, can see a number of things:

- Location of maximum shear stresses (at the maximum slopes of the membrane)
- Torque applied (volume of membrane)
- "External" corners do not add appreciability to the bending rigidity (J)

⇒ eliminate these:

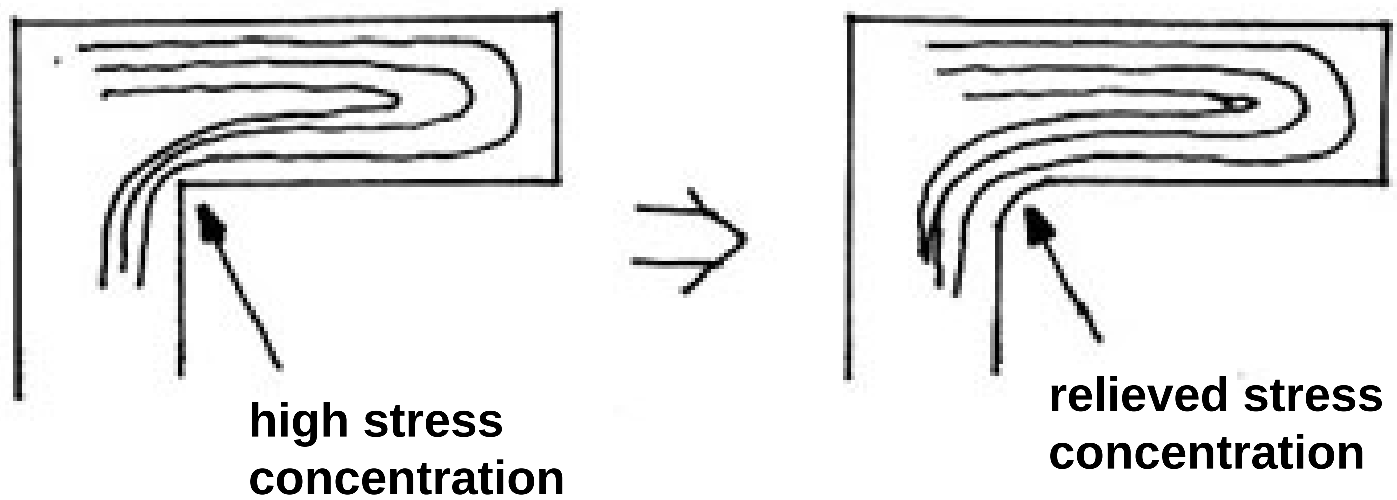
Figure 11.6€ **Representation of effect of external corners**



⇒ **<u>about</u> <u>the</u> <u>same</u>**

- Fillets (i.e. @ internal corners) eliminate stress concentrations

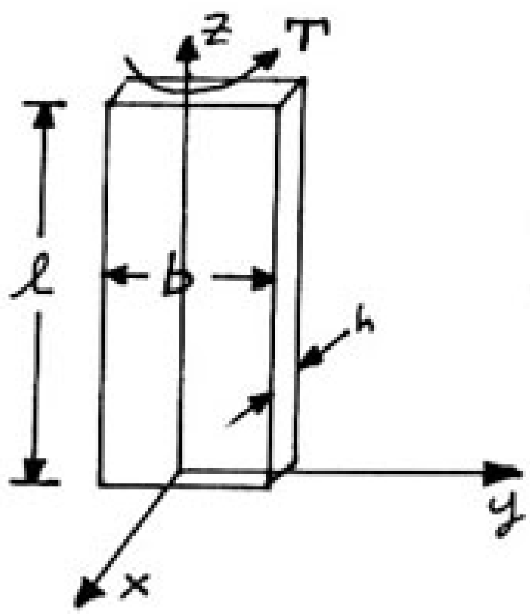
Figure 11.7 **Representation of effect of internal corners**



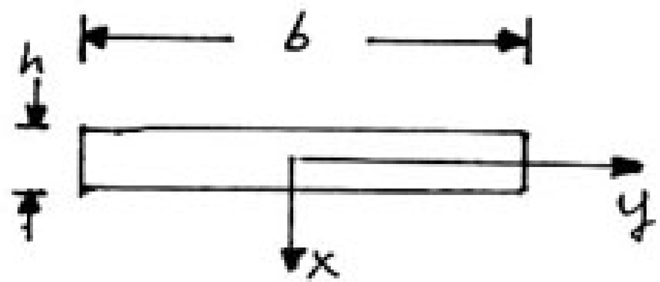
To illustrate some of these points let's consider specifically…

Torsion of a Narrow Rectangular Cross-Section

Figure 11.8 **Representation of torsion of structure with narrow rectangular cross-section**



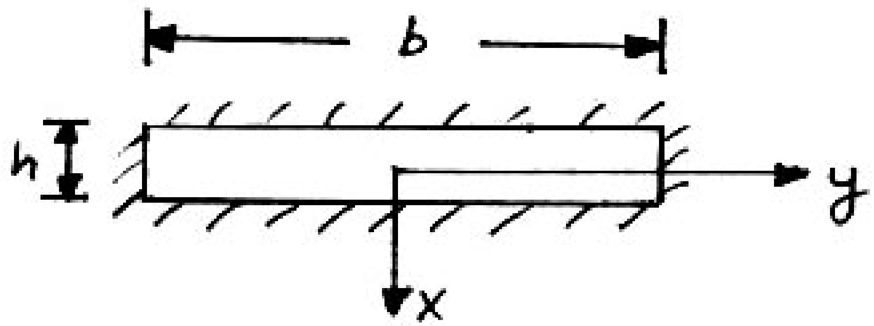
Cross-Section



b >> h

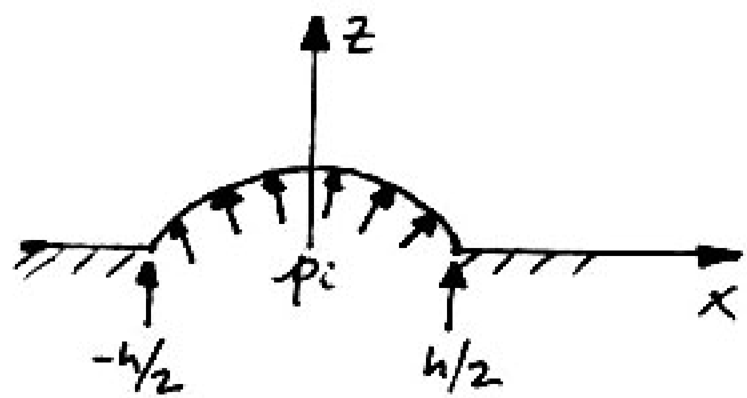
Use the Membrane Analogy for easy visualization:

Figure 11.9 **Representation of cross-section for membrane analogy**



Consider a cross-section in the middle (away from edges):

Figure 11.10 **Side view of membrane under pressure**



The governing Partial Differential Equation. is:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{p_i}{N}$$

Near the middle of the long strip (away from y = ± b/2), we would expect $\frac{\partial^2 w}{\partial y^2}$ to be small. Hence approximate via:

$$\frac{\partial^2 w}{\partial x^2} \approx -\frac{p_i}{N}$$

To get w, let's integrate:

$$\frac{\partial w}{\partial x} \approx -\frac{p_i}{N} x + C_1$$

$$w \approx -\frac{p_i}{2N} x^2 + C_1 x + C_2$$

Now apply the boundary conditions to find the constants:

$$@ \ x = +\frac{h}{2}, \quad w = 0$$

$$\Rightarrow \quad 0 = -\frac{p_i}{2N}\frac{h^2}{4} + C_1\frac{h}{2} + C_2$$

$$@\, x = -\frac{h}{2}, \quad w = 0$$

$$\Rightarrow \quad 0 = -\frac{p_i}{2N}\frac{h^2}{4} - C_1\frac{h}{2} + C_2$$

This gives:

$$C_1 = 0$$

$$C_2 = \frac{p_i h^2}{8N}$$

Thus:

$$\boxed{w \approx \frac{p_i}{2N}\left(\frac{h^2}{4} - x^2\right)}$$

Check the volume:

$$\text{Volume} = \iint w\, dx dy$$

integrating over dy:

$$= b \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{p_i}{2N} \left(\frac{h^2}{4} - x^2 \right) dx$$

$$= \frac{p_i b}{2N} \left[\frac{h^2}{4} x - \frac{x^3}{3} \right]_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$= \frac{p_i b}{2N} \left[\frac{h^2}{4} \frac{2h}{2} - \frac{2}{3} \frac{h^3}{8} \right]$$

$$\Rightarrow \text{Volume} = \frac{p_i b}{N} \frac{h^3}{12}$$

Using the Membrane Analogy:

$$p_i = -k$$

$$N = \frac{1}{2G}$$

$$\text{Volume} = -\frac{T}{2} = \frac{p_i b}{N} \frac{h^3}{12}$$

$$\frac{-k\,b\,h^3\,2\,G}{12} = -\frac{T}{2}$$

$$\Rightarrow k = -\frac{3T}{G\,b\,h^3}$$ (k - T relation)

where: $k = \frac{d\alpha}{dz}$

So:

$$\boxed{\frac{d\alpha}{dz} = \frac{T}{GJ}}$$

where: $J = \frac{bh^3}{3}$

To get the stress:

$$\sigma_{yz} = \frac{\partial w}{\partial x} = -\frac{p_i}{N}x = 2kGx$$

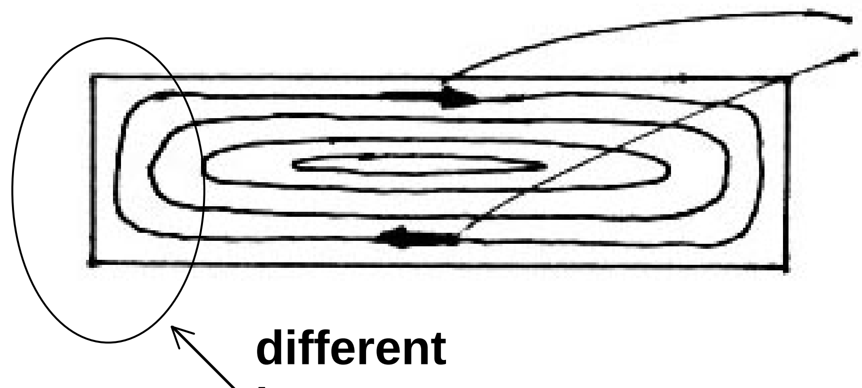
$$\boxed{\sigma_{yz} = \frac{2T}{J}x}$$

(maximum stress is twice that in a circular rod)

$$\sigma_{xz} = \frac{\partial w}{\partial y} = 0$$ **(<u>away</u> <u>from</u> <u>edges</u>)**

Near the edges, $\sigma_{xz} \neq 0$ and σ_{yz} changes:

Figure 11.11 **Representation of shear stress “flow” in narrow rectangular cross-sections**



$\sigma_{yz} = \frac{2T}{J}x$ **at these points**

(generally, these are the maximum stresses)

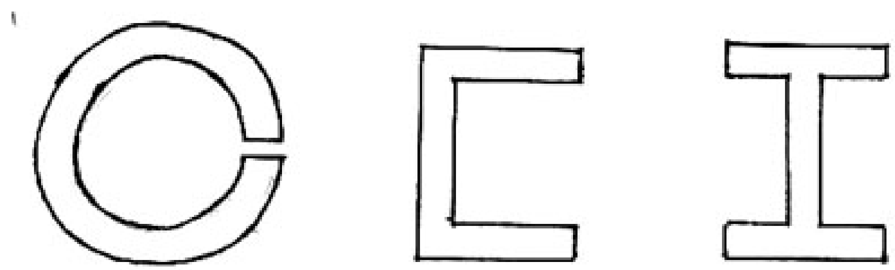
different here

Need formulae to correct for “finite” size dependent on ratio b/h. This is the key in b >> h.

Other Shapes

Through the Membrane Analogy, it can be seen that the previous theory for long, narrow rectangular sections applies also to other shapes.

Figure 11.12 **Representation of different thin open cross-sectional shapes for which membrane analogy applies**

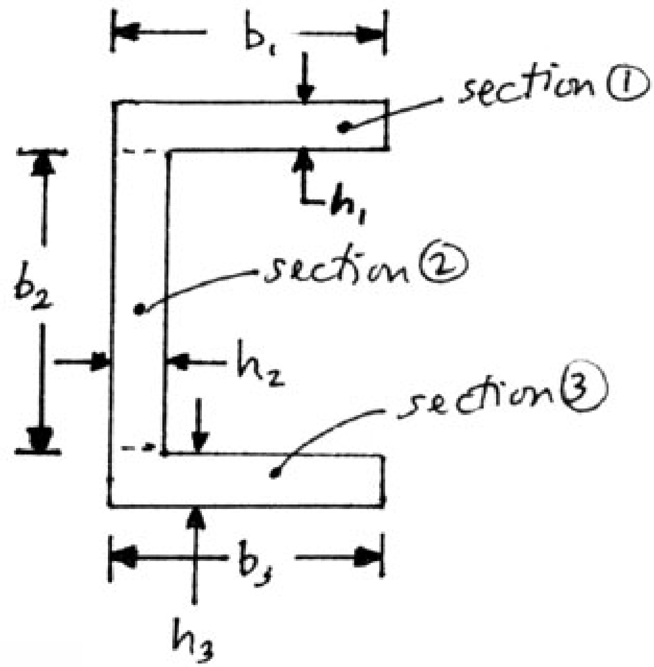


Slit tube **Channel** **I-beam**

Consider the above (as well as other similar shapes) as a long, narrow membrane

→ consider the thin channel that then results….

Figure 11.13 **Representation of generic thin channel cross-section**



$$\text{Volume} = -\frac{T}{2}$$

$$\frac{p_i}{N}\left[\frac{b_1 h_1^3}{12} + \frac{b_2 h_2^3}{12} + \frac{b_3 h_3^3}{12}\right] = -\frac{T}{2}$$

(from solution for narrow rectangle)

This gives:

$$-k2G\left[\frac{b_1h_1^3}{12} + \frac{b_2h_2^3}{12} + \frac{b_3h_3^3}{12}\right] = -\frac{T}{2}$$

$$\Rightarrow \; k = \frac{T}{GJ} \qquad \Rightarrow \text{ k - T relation}$$

where:

$$J = \frac{1}{3}b_1h_1^3 + \frac{1}{3}b_2h_2^3 + \frac{1}{3}b_3h_3^3 = \sum_i \frac{1}{3}b_ih_i^3$$

For the stresses:

$$\sigma_{yz} = \frac{\partial w}{\partial x} = -\frac{p_i}{N}x = k2Gx = \frac{2T}{J}x$$

(“local” x)

⇒ maximum

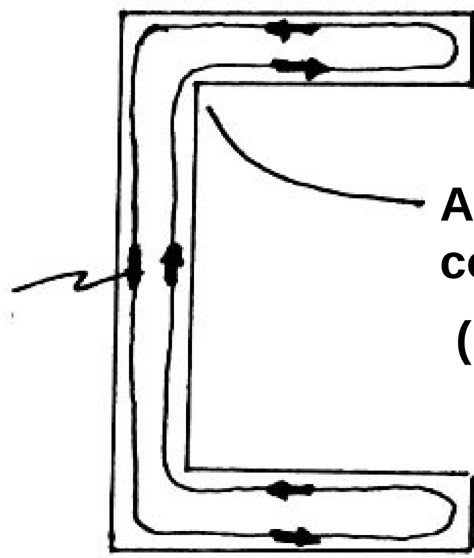
$$\sigma_{yz} = \frac{2T}{J}\frac{h_1}{2} \quad \text{in section } ①$$

$$\sigma_{yz} = \frac{2T}{J}\frac{h_2}{2} \quad \text{in section } ②$$

$$\sigma_{yz} = \frac{2T}{J}\frac{h_3}{2} \quad \text{in section } ③$$

Figure 11.14 **Representation of shear stress "flow" in thin channel under torsion**

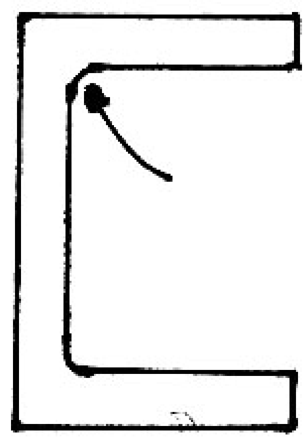
$$\sigma_{xz} = \frac{2T}{J}\,\frac{h_2}{2}$$



Actually have shear concentrations at corners (large slopes $\frac{\partial w}{\partial y}, \frac{\partial w}{\partial x}$ **)**

⇒ make "fillets" there

Figure 11.15 **Channel cross-section with "fillets" at inner corners**

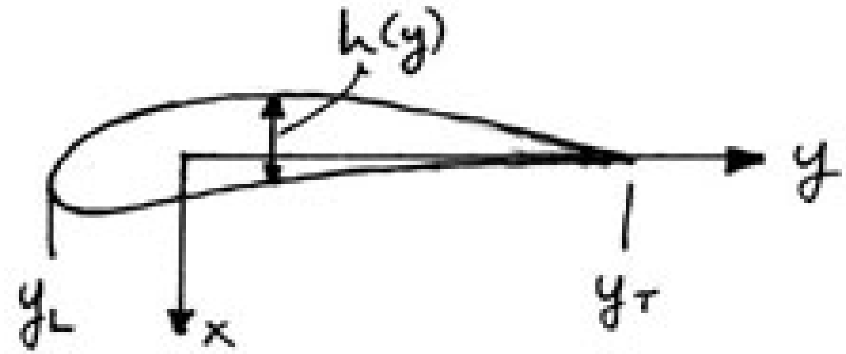


ase slope

Use the Membrane Analogy for other cross-sections

for example: variable thickness (thin) cross-section

Figure 11.15 **Representation of wing cross-section (variable thickness thin cross-section)**



Using the Membrane Analogy:

$$J \approx \frac{1}{3}\int_{y_L}^{y_T} h^3 dy \qquad \sigma_{zy} \approx \frac{2T}{J}\frac{h}{2} \qquad \text{etc.}$$

Now that we've looked at open, walled sections; let's consider closed (hollow) sections. (thick, then thin)