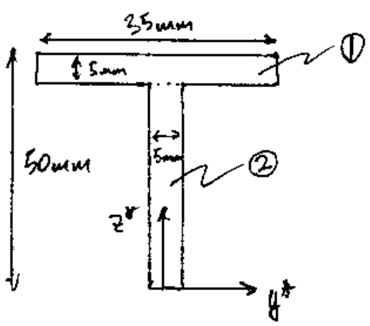
Solutions to Home Assignment #8

Warm-Up Exercises

In order to calculate the bending moments, we need to divide the cross-sections into rectangular sections. Selection of the sections does not affect the results. These properties are always determined about the centroid of the section, so our first step is to locate the centroid. We place the temporary axes in a convenient location (e.g. some corner of cross-section).



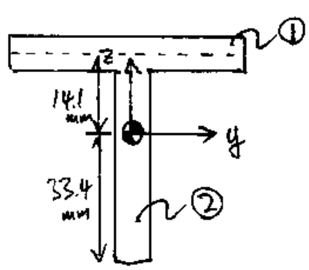
$$\bar{Y} = \frac{\Sigma A_i \bar{y}_i}{\Sigma A_i} = 0$$

$$\bar{Z} = \frac{\Sigma A_i \bar{z}_i}{\Sigma A_i} = 33.4 \text{ mm}$$

Values to calculate $\bar{Y}$ and $\bar{Z}$ are obtained from table below

Section	A_i (mm^2)	$\bar{z}_i$ (mm)	$A_i \bar{z}_i$ (mm^3)	$\bar{y}_i$ (mm)	$A_i \bar{y}_i$ (mm^3)
①	175	47.5	8310	0	0
②	225	22.5	5060	0	0
Σ	400	–	13370	–	0

Moving the axes to the centroid, we can now calculate the moments of inertia using



$$I_y = \sum_i (I_{yi} + A_i z_i^2)$$

$$I_z = \sum_i (I_{zi} + A_i y_i^2)$$

$$I_{yz} = \sum_i (I_{yzi} + A_i y_i z_i)$$

$I_{yzi} = 0$ because each section is symmetric about its own axes.

I_{yi} and I_{zi} are $\frac{bh^3}{12}$ where b is measured parallel to the axes (y for I_y, z for I_z) and h is perpendicular to the axes.

$A_i z_i^2$, $A_i y_i^2$ and $A_i y_i z_i$ comes from the parallel axis theorem.

I_y :

section	b_i (mm)	h_i (mm)	I_{yi} (mm^4)	z_i (mm)	A_i (mm^2)	$A_i z_i^2$ (mm^4)
①	35	5	365	14.1	175	34800
②	5	45	38000	-10.9	225	26700
Σ	—	—	38365	—	—	61500

$$\therefore \boxed{I_y = 99800 \text{ mm}^4}$$

I_z :

section	b_i (mm)	h_i (mm)	I_{zi} (mm^4)	y_i (mm)	A_i (mm^2)	$A_i y_i^2$ (mm^4)
①	5	35	17900	0	175	0
②	45	5	469	0	225	0
Σ	—	—	18400	—	—	0

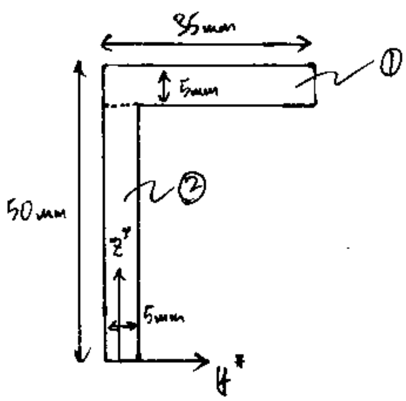
$$\therefore \boxed{I_z = 18400 \text{ mm}^4}$$

I_{yz} :

section	y_i (mm)	z_i (mm)	$A_i y_i z_i$ (mm^4)
①	0	14.1	0
②	0	-10.9	0
Σ	–	–	0

$$\therefore \boxed{I_{yz} = 0}$$

2. Again, we need to find the centroid first.



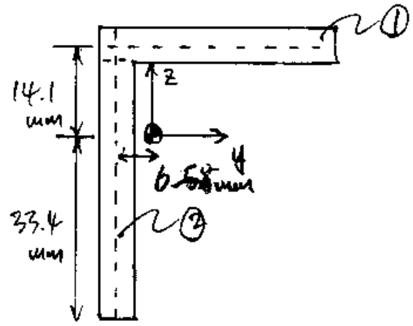
values to calculate $\bar{Y}$ and $\bar{Z}$ are obtained from table below

$$\bar{Y} = \frac{\Sigma A_i \bar{y}_i}{\Sigma A_i} = \frac{2630}{400} = 6.58\text{mm}$$

$$\bar{Z} = \frac{\Sigma A_i \bar{z}_i}{\Sigma A_i} = \frac{13370}{400} = 33.4\text{mm}$$

section	A_i (mm^2)	$\bar{z}_i$ (mm)	$A_i \bar{z}_i$ (mm^3)	y_i (mm)	$A_i \bar{y}_i$ (mm^3)
①	175	47.5	8310	15	2630
②	225	22.5	5060	0	0
Σ	400	–	13370	–	2630

*Note: location of z-direction centroid for this case is the same as prob. #1



Again, the following equations are used.

$$I_y = \sum_i (I_{yi} + A_i z_i^2)$$

$$I_z = \sum_i (I_{zi} + A_i y_i^2)$$

$$I_{yz} = \sum_i (I_{yzi} + A_i y_i z_i)$$

Note that the moment of inertia I_y for this configuration will be equal to the moment of inertia I_y for problem #1 because the geometries of the two cross-sections in the z-direction is identical. Thus,

$$\boxed{I_y = 99800 \text{ mm}^4}$$

(See first table in p2)

I_z :

section	b_i (mm)	h_i (mm)	I_{zi} (mm^4)	y_i (mm)	A_i (mm^2)	$A_i y_i^2$ (mm^4)
①	5	35	17900	8.42	175	12400
②	45	5	469	−6.58	225	9740
Σ	—	—	18400	—	—	22100

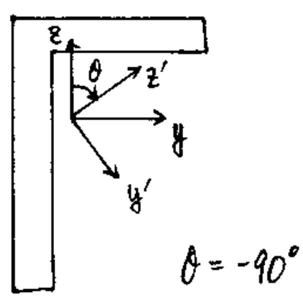
$$\therefore \boxed{I_z = 40500 \text{ mm}^4}$$

I_{yz} :

section	y_i (mm)	z_i (mm)	$A_i y_i z_i$ (mm^4)
①	8.42	14.1	20800
②	-6.58	-10.9	16100
Σ	—	—	36900

$$\therefore \boxed{I_{yz} = 36900 \text{ mm}^4}$$

3. Noting that the moments of inertia are 2nd order tensors, and that the cross-section in this problem is identical to that in problem #2 rotated 90° clockwise, we will use the transformation law to find I_y, I_z and I_{yz}.



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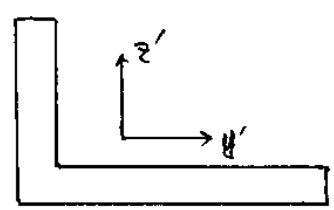


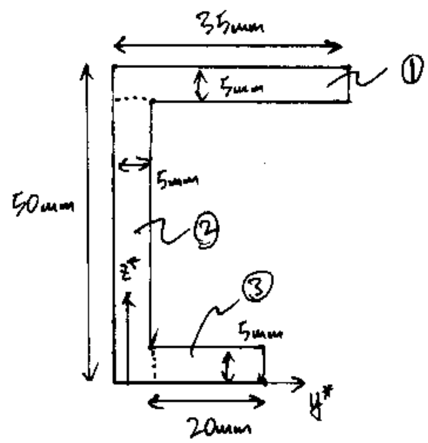
$$\begin{Bmatrix} I_{y'} \\ I_{z'} \\ I_{y'z'} \end{Bmatrix} = \begin{bmatrix} \cos^2(-90) & \sin^2(-90) & \cos(-90)\sin(-90) \\ \sin^2(-90) & \cos^2(-90) & -\cos(-90)\sin(-90) \\ -2\cos(-90)\sin(-90) & 2\cos(-90)\sin(-90) & \cos^2(-90)-\sin^2(-90) \end{bmatrix} \begin{Bmatrix} I_y \\ I_z \\ I_{yz} \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} I_{y'} \\ I_{z'} \\ I_{y'z'} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 99800 \\ 40500 \\ 36900 \end{Bmatrix}$$

$$\therefore \quad \boxed{\begin{aligned} I_{y'} &= 40500 \text{ mm}^4 \\ I_{z'} &= 99800 \text{ mm}^4 \\ I_{y'z'} &= -36900 \text{ mm}^4 \end{aligned}}$$

*Note: try obtaining these values the "conventional" way to convince yourselves.

4. Comparing this cross-section with that in problem #2, we can see that we need to add a third section at the bottom part.

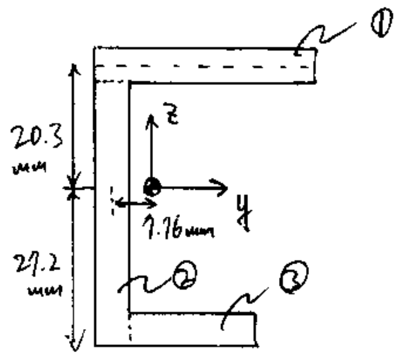


$$\bar{Y} = \frac{\Sigma A_i \bar{y}_i}{\Sigma A_i} = \frac{3880}{500} = 7.76 \text{ mm}$$

$$\bar{Z} = \frac{\Sigma A_i \bar{z}_i}{\Sigma A_i} = \frac{13620}{500} = 27.2 \text{ mm}$$

value to calculate $\bar{Y}$ and $\bar{Z}$ obtain from table below.

Section	A_i (mm²)	$\bar{z}_i$ (mm)	$A_i\bar{z}_i$ (mm³)	$\bar{y}_i$ (mm)	$A_i\bar{y}_i$ (mm³)
①+②	400	—	13370	—	2630
③	100	2.5	250	12.5	1250
Σ	500	—	13620	—	3880



$$I_y = \sum_i (I_{yi} + A_i z_i^2)$$

$$I_z = \sum_i (I_{zi} + A_i y_i^2)$$

$$I_{yz} = \sum_i (I_{yzi} + A_i y_i z_i)$$

I_y :

section	b_i(mm)	h_i(mm)	I_{yi}(mm^4)	z_i(mm)	A_i(mm^2)	$A_i z_i^2$(mm^4)
①	35	5	365	20.3	175	72100
②	5	45	38000	-4.7	225	4970
③	20	5	208	-24.7	100	61000
Σ	—	—	38600	—	—	138000

$$\therefore \boxed{I_y = 177000 \text{ mm}^4}$$

I_z :

section	b_i(mm)	h_i(mm)	I_{zi}(mm^4)	y_i(mm)	A_i(mm^2)	$A_i z_i^2$(mm^4)
①	5	35	17900	7.24	175	9170
②	45	5	469	-7.76	225	13500
③	5	20	3330	4.74	100	2250
Σ	—	—	21700	—	—	24900

$$\therefore \boxed{I_z = 46600 \text{ mm}^4}$$

I_{yz} :

section	y_i (mm)	z_i (mm)	$A_i y_i z_i$ (mm^4)
①	7.24	20.3	25700
②	-7.76	-4.7	8200
③	4.74	-24.7	-11700
Σ	—	—	22200

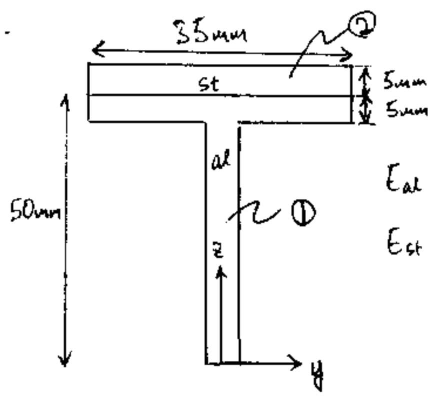
$$\therefore \boxed{I_{yz} = 22200 \text{ mm}^4}$$

For problems 5 through 7, composite beams are considered (made of aluminum and steel), so we need to use modulus-weighted properties. The products of inertia, I_{yz}^* equal to zero in all three cross-sections. Therefore, the bending stiffness in the y-axis is equal to:

$$\text{bending stiffness} = E_1 I_y^*$$

where E_1 is the reference modulus.

5.



$E_{al} = 10\,\text{Msi}$

$E_{st} = 30\,\text{Msi}$

First, we need to find the modulus-weighted centroid of cross-section.

$$\bar{z}^* = \frac{\Sigma \bar{z}_i A_i^*}{\Sigma A_i^*} = 44.3\text{ mm}$$

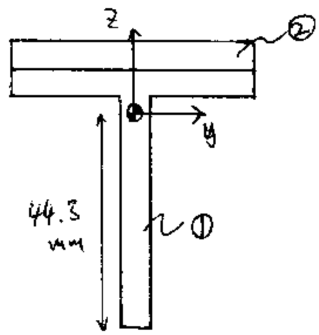
where $A_i^* = \frac{E}{E_1} A_1$. Let's take $E_1 = E_{al}$.

get section① properties from prob. #1 →

Section	A_i (mm²)	$\bar{z}_i$ (mm)	E/E_1	A_i^*	$A_i^* \bar{z}_i$ (mm³)
①	400	33.4	1	400	13360
②	175	52.5	3	525	27600
Σ	—	—	—	925	41000

* $\bar{y}^* = 0$ from symmetry.

We can now calculate the moment of inertia, I_y^*, with the axis at the modulus-weighted centroid.



$$I_y^* = \sum_i \left(I_{yi}^* + A_i^* z_i^2 \right)$$

where

$$I_{yi}^* = \frac{E}{E_1} I_{yi}$$

	section	b_i (mm)	h_i (mm)	I_{yi}^* (mm^4)	z_i (mm)	A_i^* (mm^2)	$A_i^* z_i^2$ (mm^4)
from prob. #1 →	①	—	—	99800	10.9	400	47500
	②	35	5	1090	8.2	525	35300
	Σ	—	—	10100	—	—	82800

$$\therefore I_y^* = 92900 \text{ mm}^4$$

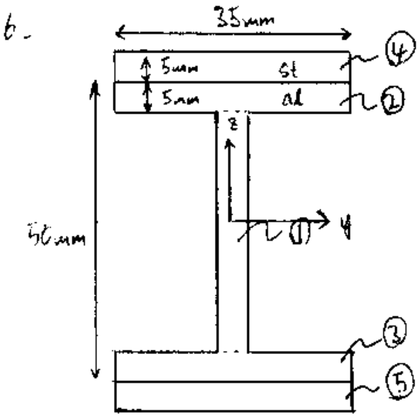
Thus, the bending stiffness is

$$* \; 1 \text{ psi} = \frac{1 \text{Lb}}{\text{in}^2}\left(\frac{1 \text{in}}{.0254 \text{m}}\right)^2\left(\frac{4.45 \text{N}}{1 \text{Lb}}\right) = 6898 \text{ Pa}$$

$$E_1 I_y^* = E_{ax} I_y^* = \overset{= 68.84 \text{GPa}}{(10 \text{Msi})}(92900 \times 10^{-9} \text{m}^4)$$

$$\Rightarrow \boxed{E_1 I_y^* = 6410 \text{ Nm}^2}$$

6.



Since this cross-section is symmetric about the y-axis, the modulus-weighted centroid coincides with the "regular" centroid and is located at the center of cross-section.

section	b_i (mm)	h_i (mm)	I_{yi}^* (mm⁴)	z_i (mm)	A_i^* (mm²)	$A_i^* z_i^2$ (mm⁴)
①	5	40	26700	0	—	0
same due to sym. ②	35	5	365	22.5	175	88600
③	35	5	365	-22.5	175	88600
same due to sym. ④	35	5	1090	27.5	525	397000
⑤	35	5	1090	-27.5	525	397000
Σ			29600			971 000

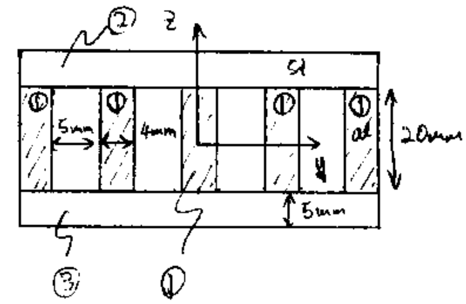
$$\therefore \quad I_y^* = \Sigma\left(I_{yi}^* + A_i^* z_i^2\right)$$

$$\Rightarrow \quad I_y^* = 1000600 \text{ mm}^4$$

$$\therefore \quad E_1 I_y^* = (68.89\,GPa)(1000600\,\text{mm}^4)$$

$$\Rightarrow \quad \boxed{E_1 I_y^* = 68900 \text{ Nm}^2}$$

7.



As in problem #6, the modulus-weighted centroid is at the center of the cross-section due to symmetry.

* Note. $I^*_{yi} = \frac{E}{E_{al}} I_{yi}$, $A^*_i = \frac{E}{E_{al}} A_i$

section	b_i (mm)	h_i (mm)	I^*_{yi} (mm^4)	z_i (mm)	A^*_i (mm^2)	$A^*_i z_i^2$ (mm^4)
① (x5)	4	20	2670	0	—	0
② (same due to sym.)	40	5	1250	12.5	600	93800
③	40	5	1250	-12.5	600	93800
Σ	—	—	15850	—	—	187600

$= 2670 \times 5 + 1250 \times 2$

$$\therefore \quad I^*_{yi} = \Sigma (I^*_{yi} + A^*_i z_i^2)$$

$$\Rightarrow \quad I^*_{yi} = 203500 \text{ mm}^4$$

$$\therefore \quad E_1 I^*_{yi} = (68.89 \text{ GPa})(203500 \text{ mm}^4)$$

$$\Rightarrow \quad \boxed{E_1 I^*_{yi} = 14000 \text{ Nm}^2}$$