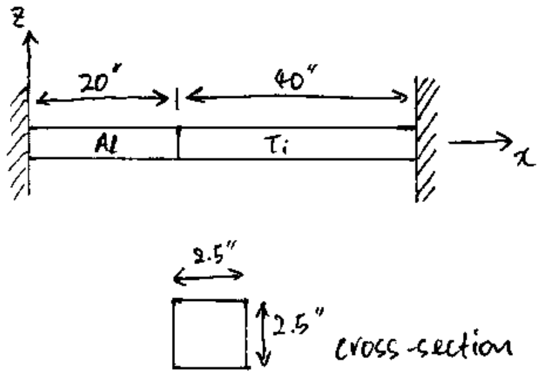
Practice Problems

4.



$\Delta T = 100°F$

Given data:

$E_{Al} = 10.0$ Msi

$\nu_{Al} = 0.30$

$\alpha_{Al} = 12.0\ \mu\varepsilon/°F$

$E_{Ti} = 15.5$ Msi

$\nu_{Ti} = 0.34$

$\alpha_{Ti} = 5.0\ \mu\varepsilon/°F$

a) Let's look at the physics of the problem first.

- Stress σ_x developed due to the temperature change will be uniform throughout the bars because of the force equilibrium requirement that the bars be in static equilibrium
- Total displacement must remain zero since both ends of the bar are clamped.

$$\Rightarrow \quad u(l) = \int_0^{l=60''} \varepsilon_x dx = 0 \qquad \text{——— ①}$$

Now let's solve the problem for the axial stress and the total strain.

The total strain in the x-direction is

$$\varepsilon_x = \varepsilon_x^M + \varepsilon_x^T$$

$$= \frac{\sigma_x}{E} + \alpha \Delta T \quad \text{——— ②}$$

Substituting equation ② into equation ①, we get

$$0 = \int_0^l \varepsilon_x \, dx = \int_0^l \left(\frac{\sigma_x}{E} + \alpha \Delta T \right) dx = \int_0^{l_1} \left(\frac{\sigma_x}{E_{Al}} + \alpha_{Al} \Delta T \right) dx + \int_{l_1}^{l} \left(\frac{\sigma_x}{E_{Ti}} + \alpha_{Ti} \Delta T \right) dx \quad \text{—③}$$

where $l_1 = 20''$. Note that σ_x is the same in the aluminum bar and in the titanium bar. Rewriting equation ③, and integrating, we get,

$$\int_0^{l_1} \left(\frac{\sigma_x}{E_{Al}} + \alpha_{Al} \Delta T \right) dx + \int_{l_1}^{l} \left(\frac{\sigma_x}{E_{Ti}} + \alpha_{Ti} \Delta T \right) dx = 0$$

$$\Rightarrow \frac{\sigma_x}{E_{Al}} l_1 + \alpha_{Al} \Delta T l_1 + \frac{\sigma_x}{E_{Ti}} (l - l_1) + \alpha_{Ti} \Delta T (l - l_1) = 0$$

$$\Rightarrow \sigma_x \left[l_1 \left(\frac{1}{E_{Al}} - \frac{1}{E_{Ti}} \right) + \frac{l}{E_{Ti}} \right] + \Delta T \left[l_1 (\alpha_{Al} - \alpha_{Ti}) + l \alpha_{Ti} \right] = 0$$

$$\therefore \sigma_x = \frac{-\Delta T \left[l_1 (\alpha_{Al} - \alpha_{Ti}) + l \alpha_{Ti} \right]}{l_1 \left(\frac{1}{E_{Al}} - \frac{1}{E_{Ti}} \right) + \frac{l}{E_{Ti}}} \quad \text{——— ③}$$

Plugging the numbers into equation ③,

$$\sigma_x = \frac{-(100°F)[(20")(12\mu s/°F - 5\mu s/°F) + (60")(5\mu s/°F)]}{(20")\left(\frac{1}{10.0\,Msi} - \frac{1}{15.5\,Msi}\right) + \frac{(60")}{15.5\,Msi}}$$

$$\therefore \quad \boxed{\sigma_x = -9.61 \text{ Ksi}}$$

Using the stress, we can get the strains.

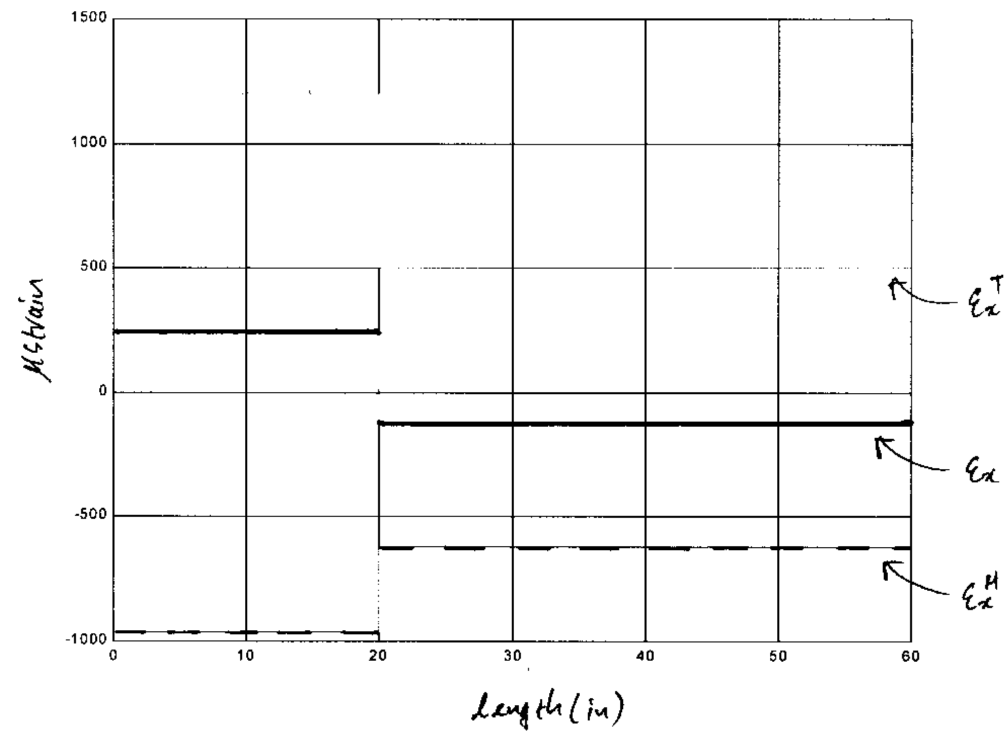
$$\varepsilon_x^T = \begin{cases} \alpha_{Al}\Delta T = (12\mu s/°F)(100°F) = 1200\mu s & 0<x<l_1 \\ \alpha_{Ti}\Delta T = (5\mu s/°F)(100°F) = 500\mu s & l_1<x<l \end{cases}$$

$$\varepsilon_x^M = \begin{cases} \frac{\sigma_x}{E_{Al}} = \frac{-9.61\,Ksi}{10.0\,Msi} = -961\,\mu s & 0<x<l_1 \\ \frac{\sigma_x}{E_{Ti}} = \frac{-9.61\,Ksi}{15.5\,Msi} = -620\,\mu s & l_1<x<l \end{cases}$$

$$\therefore \quad \boxed{\varepsilon_x = \begin{cases} 1200\mu s - 961\mu s = 239\,\mu s & 0<x<20" \\ 500\,\mu s - 620\mu s = -120\,\mu s & 20"<x<60" \end{cases}}$$

b) The three components of strain are plotted in the figure below. The results of part a) were used to generate the plot. Note that

- Thermal strain is discontinuous at $x=20"$ due to change in α.
- Mechanical strain is discontinuous at $x=20"$ due to change in E.
- Total strain is discontinuous at $x=20"$ ← this is okay as long as displacement is continuous.



c) At the new location of the material junction point, the elongation of the aluminum bar will be equal to and opposite to the elongation of the titanium bar. Together, the two elongations sum to the total elongation of the bar, which is zero. To calculate the elongations,

$$\delta_{Al} = \int_0^{20} \varepsilon_{x,Al}\, dx = (239\mu s)(20'') = 0.0048''$$

$$\delta_{Ti} = \int_{20}^{60} \varepsilon_{x,Ti}\, dx = (-120\mu s)(40'') = -0.0048''$$

equal and opposite

Thus, the displacement of the material junction point is

$$\boxed{\delta = 0.0048'' \text{ to the right}}$$

d) St. Venant's principle applies at the boundaries (ends) where $\varepsilon_y = 0$. In addition, it also applies at the junction where the two materials meet.

$$\varepsilon_{y\,Al} = \varepsilon_{y\,Ti} \quad \leftarrow \text{for joint continuity}$$

We know that St. Venant's principle is important at the joint because

$$\nu_{Al} \neq \nu_{Ti}$$

At all of these points, the material will develop biaxial stresses (σ_x and σ_y) to satisfy the boundary conditions. In addition, the BC's may cause development of shear stress σ_{xy}.

e) From parts a) through c), we know

$$\left.\begin{array}{l} \varepsilon_{x\,Al},\ \varepsilon^{T}_{x\,Al},\ \varepsilon^{M}_{x\,Al},\ \sigma_{x\,Al} \\ \varepsilon_{x\,Ti},\ \varepsilon^{T}_{x\,Ti},\ \varepsilon^{M}_{x\,Ti},\ \sigma_{x\,Ti} \end{array}\right\} \text{same},\ \sigma_x = \sigma_{x\,Al} = \sigma_{x\,Ti}$$

Now let's write out all the pertinent equations.

Compatibility says:

$$\varepsilon_{yAl} = \varepsilon_{yTi} \quad \text{——— ①}$$

$$\varepsilon_{zAl} = \varepsilon_{zTi} \quad \text{——— ②}$$

The total strains are

$$\varepsilon_{yAl} = \alpha_{Al}\Delta T + \varepsilon^{M}_{yAl} \quad \text{——— ③}$$

$$\varepsilon_{zAl} = \alpha_{Al}\Delta T + \varepsilon^{M}_{zAl} \quad \text{——— ④}$$

$$\varepsilon_{yTi} = \alpha_{Ti}\Delta T + \varepsilon^{M}_{yTi} \quad \text{——— ⑤}$$

$$\varepsilon_{zTi} = \alpha_{Ti}\Delta T + \varepsilon^{M}_{zTi} \quad \text{——— ⑥}$$

Equilibrium say:

$$\sigma_{yAl} + \sigma_{yTi} = 0 \quad \text{——— ⑦}$$

$$\sigma_{zAl} + \sigma_{zTi} = 0 \quad \text{——— ⑧}$$

Stress-strain relations:

$$\varepsilon^{M}_{yAl} = \frac{1}{E_{Al}}\left(-\nu_{Al}\sigma_x + \sigma_{yAl} - \nu_{Al}\sigma_{zAl}\right) \quad \text{——— ⑨}$$

$$\varepsilon^{M}_{zAl} = \frac{1}{E_{Al}}\left(-\nu_{Al}\sigma_x - \nu_{Al}\sigma_{yAl} + \sigma_{zAl}\right) \quad \text{——— ⑩}$$

$$\varepsilon^{M}_{yTi} = \frac{1}{E_{Ti}}\left(-\nu_{Ti}\sigma_x + \sigma_{yTi} - \nu_{Ti}\sigma_{zTi}\right) \quad \text{——— ⑪}$$

$$\varepsilon^{M}_{zTi} = \frac{1}{E_{Ti}}\left(-\nu_{Ti}\sigma_x - \nu_{Ti}\sigma_{yTi} + \sigma_{zTi}\right) \quad \text{——— ⑫}$$

Since there are 12 equations and 12 unknowns,

$$\text{unknowns}\left(\begin{array}{cccccc} \sigma_{yAl} & \sigma_{zAl} & \varepsilon^M_{yAl} & \varepsilon^M_{zAl} & \varepsilon_{yAl} & \varepsilon_{zAl} \\ \sigma_{yTi} & \sigma_{zTi} & \varepsilon^M_{yTi} & \varepsilon^M_{zTi} & \varepsilon_{yTi} & \varepsilon_{zTi} \end{array}\right.$$

we can solve the problem. The most straightforward way is to use the two compatibility equations, ① and ②.

- $\varepsilon_{yAl} = \varepsilon_{yTi}$:

③,⑤ $$\Rightarrow \quad \alpha_{Al}\Delta T + \varepsilon^M_{yAl} = \alpha_{Ti}\Delta T + \varepsilon^M_{yTi}$$

⑨, ⑪ $$\Rightarrow \quad \alpha_{Al}\Delta T + \frac{1}{E_{Al}}\left(-\nu_{Al}\sigma_x + \sigma_{yAl} - \nu_{Al}\sigma_{zAl}\right)$$

$$= \alpha_{Ti}\Delta T + \frac{1}{E_{Ti}}\left(-\nu_{Ti}\sigma_x + \sigma_{yTi} - \nu_{Ti}\sigma_{zTi}\right)$$

⑦,⑧ $$\Rightarrow \quad \alpha_{Al}\Delta T + \frac{1}{E_{Al}}\left(-\nu_{Al}\sigma_x + \sigma_{yAl} - \nu_{Al}\sigma_{zAl}\right)$$

$$= \alpha_{Ti}\Delta T + \frac{1}{E_{Ti}}\left(-\nu_{Ti}\sigma_x - \sigma_{yAl} + \nu_{Ti}\sigma_{zAl}\right)$$

$$\Rightarrow \quad \left(\frac{1}{E_{Al}} + \frac{1}{E_{Ti}}\right)\sigma_{yAl} - \left(\frac{\nu_{Al}}{E_{Al}} + \frac{\nu_{Ti}}{E_{Ti}}\right)\sigma_{zAl}$$

$$= (\alpha_{Ti} - \alpha_{Al})\Delta T + \left(\frac{\nu_{Al}}{E_{Al}} - \frac{\nu_{Ti}}{E_{Ti}}\right)\sigma_x \quad \text{— ⑬}$$

- $\varepsilon_{zAl} = \varepsilon_{zTi}$:

$$\Rightarrow \quad \alpha_{Al}\Delta T + \varepsilon^M_{zAl} = \alpha_{Ti}\Delta T + \varepsilon^M_{zTi}$$

$$\Rightarrow \quad \alpha_{Al}\Delta T + \frac{1}{E_{Al}}\left(-\nu_{Al}\sigma_x - \nu_{Al}\sigma_{yAl} + \sigma_{zAl}\right)$$

$$= \alpha_{Ti}\Delta T + \frac{1}{E_{Ti}}\left(-\nu_{Ti}\sigma_x - \nu_{Ti}\sigma_{yTi} + \sigma_{zTi}\right)$$

$$\Rightarrow \quad \alpha_{Al}\Delta T + \frac{1}{E_{Al}}\left(-\nu_{Al}\sigma_x - \nu_{Al}\sigma_{yAl} + \sigma_{zAl}\right)$$

$$= \alpha_{Ti}\Delta T + \frac{1}{E_{Ti}}\left(-\nu_{Ti}\sigma_x + \nu_{Ti}\sigma_{yAl} - \sigma_{zAl}\right)$$

$$\Rightarrow \quad -\left(\frac{\nu_{Al}}{E_{Al}} + \frac{\nu_{Ti}}{E_{Ti}}\right)\sigma_{yAl} + \left(\frac{1}{E_{Al}} + \frac{1}{E_{Ti}}\right)\sigma_{zAl}$$

$$= (\alpha_{Ti} - \alpha_{Al})\Delta T + \left(\frac{\nu_{Al}}{E_{Al}} - \frac{\nu_{Ti}}{E_{Ti}}\right)\sigma_x$$

Solving equations ⑬ and ⑭ simultaneously, we get,

- $\left(\frac{1}{E_{Al}} + \frac{1}{E_{Ti}}\right) \times$ ⑬

$$\Rightarrow \quad \left(\frac{1}{E_{Al}} + \frac{1}{E_{Ti}}\right)^2 \sigma_{yAl} - \left(\frac{1}{E_{Al}} + \frac{1}{E_{Ti}}\right)\left(\frac{\nu_{Al}}{E_{Al}} + \frac{\nu_{Ti}}{E_{Ti}}\right)\sigma_{zAl}$$

$$= \left(\frac{1}{E_{Al}} + \frac{1}{E_{Ti}}\right)\left[(\alpha_{Ti} - \alpha_{Al})\Delta T + \left(\frac{\nu_{Al}}{E_{Al}} - \frac{\nu_{Ti}}{E_{Ti}}\right)\sigma_x\right.$$

- $\left(\frac{\nu_{Al}}{E_{Al}} + \frac{\nu_{Ti}}{E_{Ti}}\right) \times$ ⑭

$$\Rightarrow \quad -\left(\frac{\nu_{Al}}{E_{Al}} + \frac{\nu_{Ti}}{E_{Ti}}\right)^2 \sigma_{yAl} + \left(\frac{1}{E_{Al}} + \frac{1}{E_{Ti}}\right)\left(\frac{\nu_{Al}}{E} + \frac{\nu_{Ti}}{E_{Ti}}\right)\sigma_{zAl}$$

$$= \left(\frac{\nu_{Al}}{E_{Al}} + \frac{\nu_{Ti}}{E_{Ti}}\right)\left[(\alpha_{Ti} - \alpha_{Al})\Delta T + \left(\frac{\nu_{Al}}{E_{Al}} - \frac{\nu_{Ti}}{E_{Ti}}\right.\right.$$

Adding equations ⑪ and ⑮ and rearranging,

$$\sigma_{yAl} = \frac{\left(\frac{1+\nu_{Al}}{E_{Al}} + \frac{1+\nu_{Ti}}{E_{Ti}}\right)\left[(\alpha_{Ti} - \alpha_{Al})\Delta T + \left(\frac{\nu_{Al}}{E_{Al}} - \frac{\nu_{Ti}}{E_{Ti}}\right)\sigma_x\right]}{\left(\frac{1}{E_{Al}} + \frac{1}{E_{Ti}}\right)^2 - \left(\frac{\nu_{Al}}{E_{Al}} + \frac{\nu_{Ti}}{E_{Ti}}\right)^2}$$

Note that if we substract equation ⑭ from equation ⑬, we get,

$$\left(\frac{1}{E_{Al}} + \frac{1}{E_{Ti}}\right)\sigma_{yAl} + \left(\frac{\nu_{Al}}{E_{Al}} + \frac{\nu_{Ti}}{E_{Ti}}\right)\sigma_{yAl}$$

$$- \left(\frac{\nu_{Al}}{E_{Al}} + \frac{\nu_{Ti}}{E_{Ti}}\right)\sigma_{zAl} \quad \left(\frac{1}{E_{Al}} + \frac{1}{E_{Ti}}\right)\sigma_{zAl} = 0$$

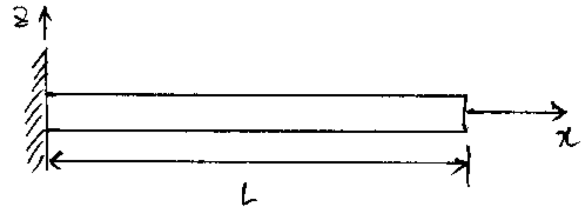
$$\Rightarrow \quad \sigma_{zAl} = \sigma_{yAl} \qquad \text{———— ⑱}$$

This relationship also makes sence from an intuitive point of view. Both directions are unconstrained and have the same dimensions (see cross-section), so the stresses should be the same. Thus, the stresses are

$$\boxed{\begin{aligned} \sigma_{yAl} = \sigma_{zAl} &= \frac{\left(\frac{1+\nu_{Al}}{E_{Al}} + \frac{1+\nu_{Ti}}{E_{Ti}}\right)\left[(\alpha_{Ti} - \alpha_{Al})\Delta T + \left(\frac{\nu_{Al}}{E_{Al}} - \frac{\nu_{Ti}}{E_{Ti}}\right)\sigma_x\right]}{\left(\frac{1}{E_{Al}} + \frac{1}{E_{Ti}}\right)^2 - \left(\frac{\nu_{Al}}{E_{Al}} + \frac{\nu_{Ti}}{E_{Ti}}\right)^2} \\ \sigma_{yTi} = \sigma_{zTi} &= -\frac{\left(\frac{1+\nu_{Al}}{E_{Al}} + \frac{1+\nu_{Ti}}{E_{Ti}}\right)\left[(\alpha_{Ti} - \alpha_{Al})\Delta T + \left(\frac{\nu_{Al}}{E_{Al}} - \frac{\nu_{Ti}}{E_{Ti}}\right)\sigma_x\right]}{\left(\frac{1}{E_{Al}} + \frac{1}{E_{Ti}}\right)^2 - \left(\frac{\nu_{Al}}{E_{Al}} + \frac{\nu_{Ti}}{E_{Ti}}\right)^2} \end{aligned}}$$

The strains (and the components) can be obtained by plugging the stresses into equations ③ ~ ⑥ and ⑨ ~ ⑫.

5.



$$T(x) = T_{ref} + T_0\left(\frac{x}{L}\right)$$

Given data:

$E_L = 140\,GPa$

$\nu_{LT} = 0.28$

$E_T = 20.0\,GPa$

$\nu_{TL} = 0.040$

$\alpha_L = -0.11\,\mu s/°C$

$\alpha_T = 6.7\,\mu s/°C$

Since the temperature distribution varies in the x-direction, the bar/beam is going to deflect in the x-direction only. The bar/beam is unconstrained at $x = L$, so this mea s that there will be no force in the x-direction. The the mechanical stress is zero.

$$\text{No force} \Rightarrow \sigma_x = 0 \Rightarrow \varepsilon_x^M = 0.$$

The total strain consists of only the thermal strain.

$$\varepsilon_x = \varepsilon_x^T + \cancelto{0}{\varepsilon_x^M} = \varepsilon_x^T \qquad \text{———} \ ①$$

$$\Rightarrow \quad \varepsilon_x = \varepsilon_x^T = \alpha_L \Delta T = \alpha_L (T - T_{ref}) \quad \text{——— ②}$$

Using the given data, equation ② becomes

$$\varepsilon_x = (-0.11 \mu\varepsilon/^\circ C) T_0 \left(\frac{x}{L}\right) \quad \text{——— ③}$$

Rewriting equation ○ in terms of the tip temperature, T_{tip}, we get,

$* \; T_{tip} = T(L) = T_{ref} + T_0$

$\Rightarrow T_0 = T_{tip} - T_{ref}$

$$\varepsilon_x = (-0.11 \mu\varepsilon/^\circ C)(T_{tip} - T_{ref}) \left(\frac{x}{L}\right) \quad \text{——— ④}$$

The tip deflection (= total elongation) is

$$\Delta L = \int_0^L \varepsilon_x \, dx = \int_0^L (-0.11 \mu\varepsilon/^\circ C)(T_{tip} - T_{ref}\left(\frac{x}{L}\right) dx$$

$$= (-0.11 \mu\varepsilon/^\circ C)(T_{tip} - T_{ref}) \frac{x^2}{2L} \Big|_0^L$$

$$\therefore \quad \boxed{\Delta L = (-0.11 \times 10^{-6}\ 1/^\circ C)(T_{tip} - T_{ref}) \frac{L}{2}}$$