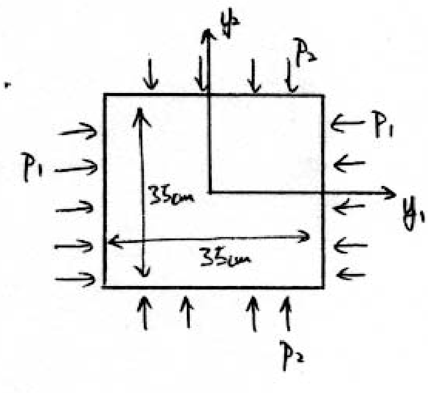
Practice Problems

4. 2-meter long aluminum bar with 35 cm x 35 cm uniform cross-section.



Material properties:

$\nu = 0.3$

$E = 70.8\,GPa$.

Let's compute the plane stress and plane strain solutions and compare the stresses and strains.

Plane stress solution

$$\left.\begin{aligned} \sigma_{11} &= -p_1 \\ \sigma_{22} &= -p_2 \\ \sigma_{12} &= 0 \end{aligned}\right\} \text{given}$$

* pressure gives compression stress.

$$\left.\begin{aligned} \sigma_{13} &= 0 \\ \sigma_{23} &= 0 \\ \sigma_{33} &= 0 \end{aligned}\right\} \text{plane stress assumption}$$

$$\varepsilon_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{33} = \frac{1}{E}(-p_1 + \nu p_2)$$

$$\Rightarrow \quad \varepsilon_{11} = \frac{1}{E}(-p_1 + \nu p_2) \qquad \text{———} \quad ①$$

$$\varepsilon_{22} = -\frac{\nu}{E}\sigma_{11} + \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{33} = \frac{1}{E}(\nu p_1 - p_2)$$

$$\Rightarrow \quad \varepsilon_{22} = \frac{1}{E}(\nu p_1 - p_2) \qquad \text{——— ②}$$

$$\varepsilon_{33} = -\frac{\nu}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} + \frac{1}{E}\sigma_{33} = \frac{\nu}{E}(p_1 + p_2)$$

$$\Rightarrow \quad \varepsilon_{33} = \frac{\nu}{E}(p_1 + p_2) \qquad \text{——— ③}$$

$$\varepsilon_{23} = \frac{1}{2G}\sigma_{23} = 0 \qquad \text{——— ④}$$

$$\varepsilon_{13} = \frac{1}{2G}\sigma_{13} = 0 \qquad \text{——— ⑤}$$

$$\varepsilon_{12} = \frac{1}{2G}\sigma_{12} = 0 \qquad \text{——— ⑥}$$

<u>Plane strain solution</u>

$$\left.\begin{aligned} \sigma_{11} &= -p_1 \\ \sigma_{22} &= -p_2 \\ \sigma_{12} &= 0 \end{aligned}\right\} \text{ given} \qquad \left.\begin{aligned} \varepsilon_{13} &= 0 \\ \varepsilon_{23} &= 0 \\ \varepsilon_{33} &= 0 \end{aligned}\right\} \text{ plane strain assumption}$$

$$\varepsilon_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{33} \qquad \text{——— ⑦}$$

$$\varepsilon_{22} = -\frac{\nu}{E}\sigma_{11} + \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{33} \qquad \text{——— ⑧}$$

$$\varepsilon_{33} = -\frac{\nu}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} + \frac{1}{E}\sigma_{33} \tag{9}$$

$$\varepsilon_{23} = \frac{1}{2G}\sigma_{23} \tag{10}$$

$$\varepsilon_{13} = \frac{1}{2G}\sigma_{13} \tag{11}$$

$$\varepsilon_{12} = \frac{1}{2G}\sigma_{12} \tag{12}$$

From the plane strain assumption, $\varepsilon_{23} = \varepsilon_{13} = 0$, we know that

$$\sigma_{23} = \sigma_{13} = 0$$

in equation (10) and (11). From the given applied stress, $\sigma_{12} = 0$,

$$\varepsilon_{12} = 0$$

from equation (12). From the plane strain assumption, $\varepsilon_{33} = 0$, we can get σ_{33} as follows. Equation (9) :

$$\varepsilon_{33} = -\frac{\nu}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} + \frac{1}{E}\sigma_{33} = 0$$

$$\Rightarrow \quad \sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \tag{13}$$

Substituting the applied stress into equation (13), we get

$$\sigma_{33} = -\nu(p_1 + p_2) \tag{14}$$

Using equation (14) and substituting into equations (7) and (8), we get

$$\varepsilon_{11} = \frac{1+\nu}{E}\left[-(1-\nu)p_1 + \nu p_2\right] \qquad (15)$$

$$\varepsilon_{22} = \frac{1+\nu}{E}\left[\nu p_1 - (1-\nu)p_2\right] \qquad (16)$$

Summarizing,

Plane stress:

$\sigma_{11} = -p_1$	$\varepsilon_{11} = \frac{1}{E}(-p_1 + \nu p_2)$
$\sigma_{22} = -p_2$	$\varepsilon_{22} = \frac{1}{E}(\nu p_1 - p_2)$
$\sigma_{33} = 0$	$\varepsilon_{33} = \frac{\nu}{E}(p_1 + p_2)$
$\sigma_{23} = 0$	$\varepsilon_{23} = 0$
$\sigma_{13} = 0$	$\varepsilon_{13} = 0$
$\sigma_{12} = 0$	$\varepsilon_{12} = 0$

Plane strain:

$$\sigma_{11} = -p_1 \qquad \varepsilon_{11} = \frac{1+\nu}{E}\left[-(1-\nu)p_1 + \nu p_2\right]$$

$$\sigma_{22} = -p_2 \qquad \varepsilon_{22} = \frac{1+\nu}{E}\left[\nu p_1 - (1-\nu)p_2\right]$$

$$\sigma_{33} = -\nu(p_1+p_2) \qquad \varepsilon_{33} = 0$$

$$\sigma_{23} = 0 \qquad \varepsilon_{23} = 0$$

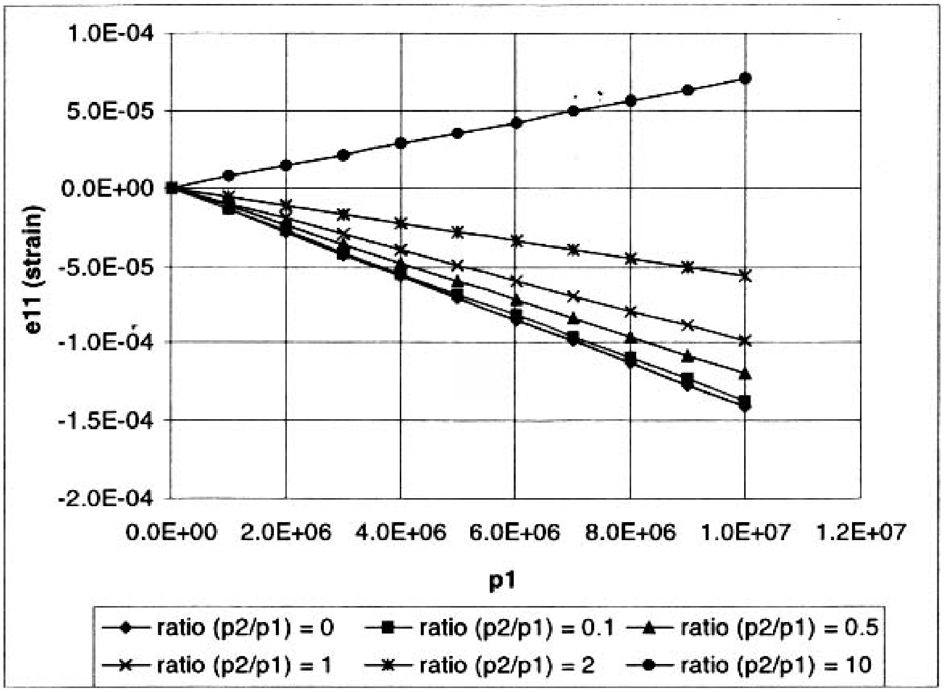
$$\sigma_{13} = 0 \qquad \varepsilon_{13} = 0$$

$$\sigma_{12} = 0 \qquad \varepsilon_{12} = 0$$

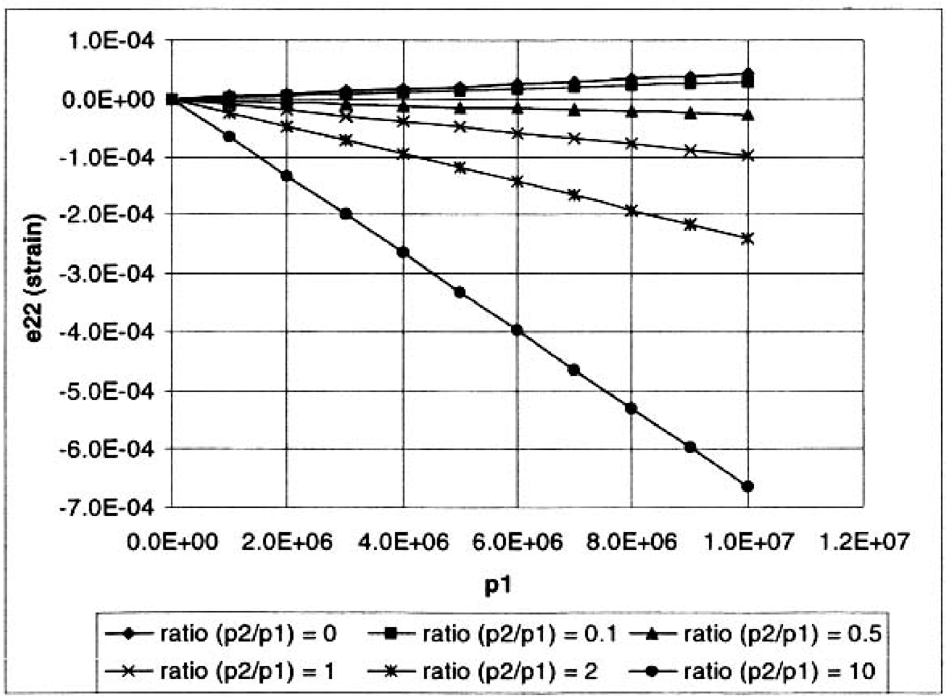
Notice that our plane stress and plane strain assumptions produce different stress and strain results.

Now, let's draw plots of strains ε_{11}, ε_{22}, and ε_{33} for various ratios and values of the two pressures, p_1 and p_2.

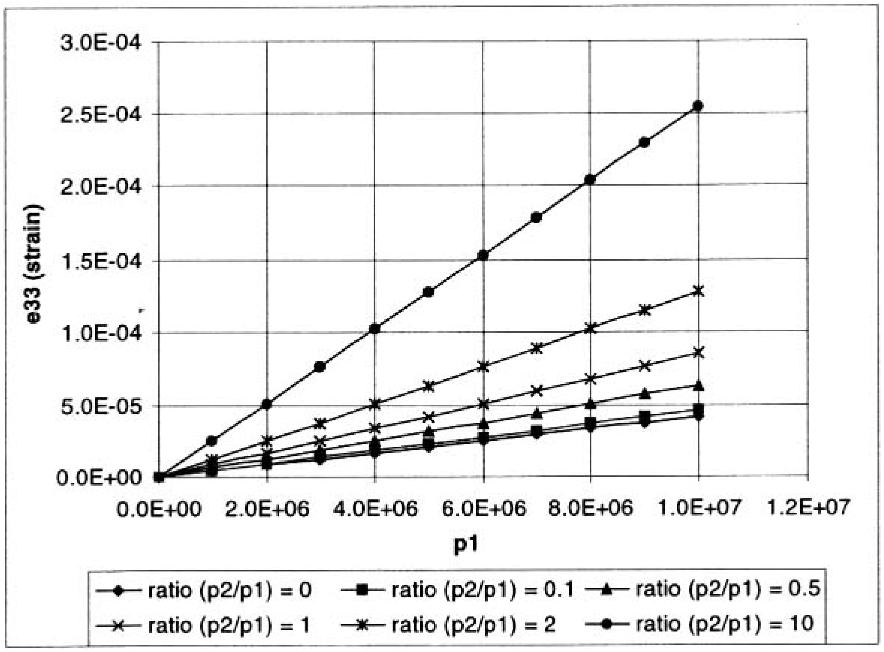
Plane Stress Plots.



Plot 1



Plot 2



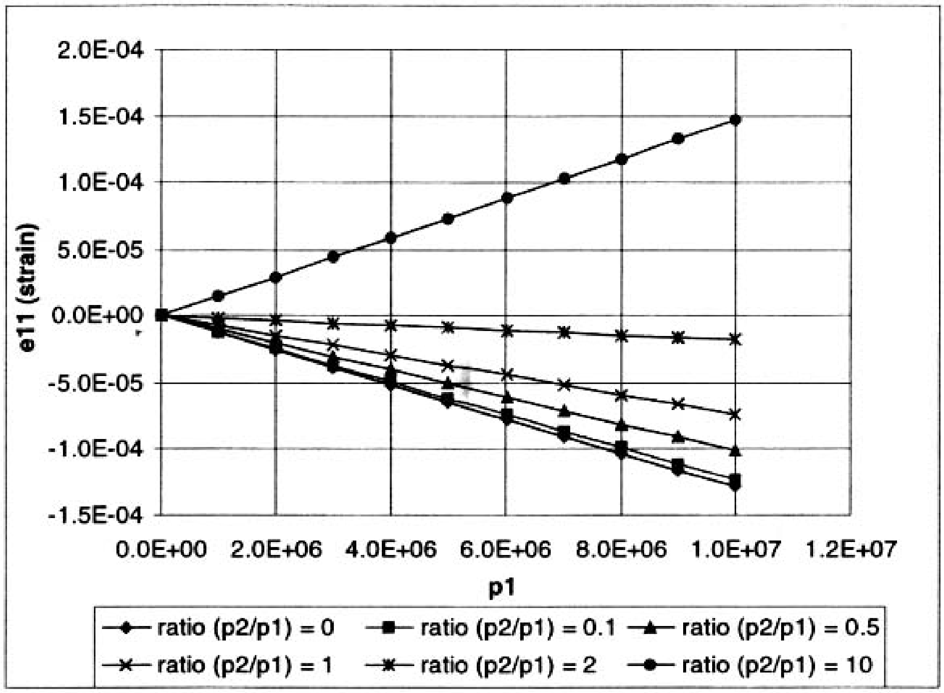
Plot 3

From the plots, we can see that :

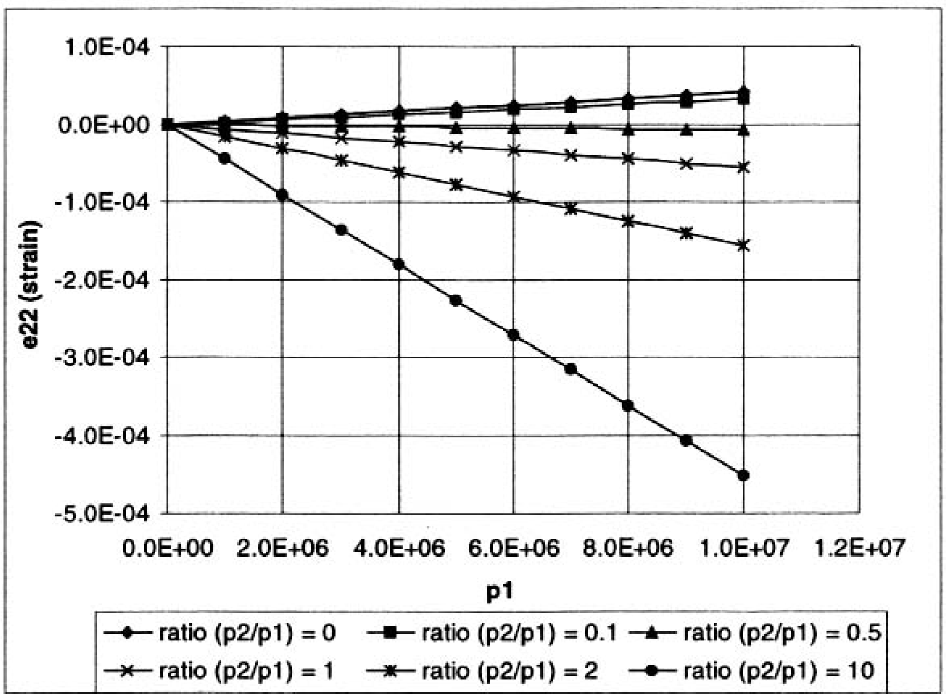
- For $P_2/P_1 = 0$, ε_{11} is always negative and ε_{22} is always positive. As the ratio of P_2/P_1 increases, ε_{11} also increases and becomes positive for $P_2/P_2 > \frac{1}{\underset{=\nu}{0.3}}$. Similarly, ε_{22} becomes negative for $P_2/P_1 > \underset{\nu}{\underset{\parallel}{0.3}}$

- ε_{33} is positive for all ratios of P_2/P_1. It also increases with increasing P_2/P_1 and P_1.

Plane Strain Plots



Plot 4



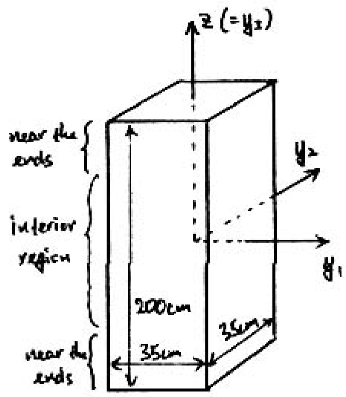
Plot 5

From the plots, we can see that

- For $P_2/P_1 = 0$, ε_{11} is always negative and ε_{22} is always positive. As the radio of P_2/P_1 increases, ε_{11} also increasing and becomes positive for $P_2/P_1 > \frac{(1-0.3)}{0.3}$. Similarly, ε_{22} becomes negative for $P_2/P_1 > \frac{0.3}{1-0.3}$.

- ε_{33} is zero.

What is the applicability of the two models?



(not to scale)

<u>Near the ends</u>

From the diagram, it can be seen that at the ends of the bar $\sigma_{33} = 0$ because there is no load acting at the ends. Also, the shear stresses, $\sigma_{13} = \sigma_{23} = 0$ because there are no shear load at the ends. Therefore, in the region near the ends, we can assumed that a state of plane stress exists. Note that since there are no applied loads at the ends and no geometric boundary conditions, there will be expansion in the z-direction. The positive ε_{33} in Plot 3 results from this expansion. The plane strain state does not describe the stress and strain state in this region, and therefore, does not apply.

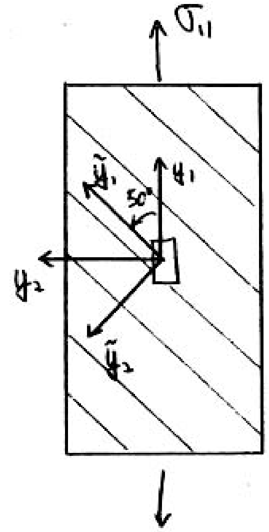
<u>Interior region</u>

In the region near the center of the bar, the effects of $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$ at the ends of the bar will not be precent. Instead, there is a lot of material on either sides of the bar that inhibits this region from expanding in the z-direction. The conditions are $\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$. Since the expansion in the z-direction is restricted, stress in this direction, σ_{zz}, must arise. Therefore, in the interior region, we can assume a state of plane strain.

* Note : A transition region between the interior region and the regions near the ends exist, which cannot be modeled accurately using either plane strain or plane stress. We can, however, infer

a gradient of stresses and strains between the two idealized regions from the plane strain and plane stress assumptions.

5. The following information is given.



Applied Load:

$\sigma_{11} = 200 MPa$

$(\sigma_{22} = \sigma_{12} = 0)$

Material properties:

$E_L = 143\ GPa$

$E_T = 9.8\ GPa$

$G_{LT} = 6.0\ GPa$

$\nu_{LT} = 0.28$

(a) We need to rotate the axes to the material (or fiber) axes. The transformation for tensors is:

$$\hat{\sigma}_{\alpha\beta} = l_{\alpha\sigma}\, l_{\beta\gamma}\, \sigma_{\sigma\gamma} \qquad \text{①}$$

↑ stress in $\tilde{y}_1 - \tilde{y}_2$ axes ↑ stress in $y_1 - y_2$ axes

Equation ① can be written out in matrix form as follows.

$$\begin{Bmatrix} \tilde{\sigma}_{11} \\ \tilde{\sigma}_{22} \\ \tilde{\sigma}_{12} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \quad \text{--- } ②$$

Plugging the applied load and $\theta = 50^\circ$ into equation ②, we get

$$\begin{Bmatrix} \tilde{\sigma}_{11} \\ \tilde{\sigma}_{22} \\ \tilde{\sigma}_{12} \end{Bmatrix} = \begin{bmatrix} 0.4132 & 0.5868 & 0.9848 \\ 0.5868 & 0.4132 & -0.9848 \\ -0.4924 & 0.4924 & -0.1736 \end{bmatrix} \begin{Bmatrix} 200\,MPa \\ 0 \\ 0 \end{Bmatrix} \quad \text{--- } ③$$

$$\Rightarrow \boxed{\begin{aligned} \hat{\sigma}_{11} &= 82.6\ MPa \\ \hat{\sigma}_{22} &= 117.0\ MPa \\ \hat{\sigma}_{12} &= -98.5\ MPa \end{aligned}}$$

(b) To get the strains, we use the stress-strain relations with the given material properties. The 2-D orthotropic stress-strain relations are:

$$\begin{Bmatrix} \hat{\varepsilon}_{11} \\ \hat{\varepsilon}_{22} \\ \hat{\varepsilon}_{33} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_L} & \overset{= -\frac{\nu_{TL}}{E_T}}{-\frac{\nu_{LT}}{E_L}} & 0 \\ -\frac{\nu_{LT}}{E_L} & \frac{1}{E_T} & 0 \\ 0 & 0 & \frac{1}{2G_{LT}} \end{bmatrix} \begin{Bmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \\ \hat{\sigma}_{12} \end{Bmatrix} \quad \text{--- } ④$$

Plugging the given material properties and stresses from (a) into equation ④, we get:

$$\tilde{\varepsilon}_{11} = \frac{1}{143GPa}(82.6MPa) - \frac{0.28}{143GPa}(117.4MPa)$$

$$\Rightarrow \boxed{\tilde{\varepsilon}_{11} = 348 \times 10^{-6}}$$

$$\tilde{\varepsilon}_{22} = -\frac{0.28}{143GPa}(82.6MPa) + \frac{1}{9.8GPa}(117.4MPa)$$

$$\Rightarrow \boxed{\tilde{\varepsilon}_{22} = 11820 \times 10^{-6}}$$

$$\tilde{\varepsilon}_{12} = \frac{1}{2 \times (6.0GPa)}(-98.5MPa)$$

$$\Rightarrow \boxed{\hat{\varepsilon}_{12} = -8210 \times 10^{-6}}$$

(c) Now, we need to transform the strains in the $\tilde{y}_1-\tilde{y}_2$ axes back into the strains in the y_1-y_2 axes. Again, the the transformation for tensors is:

$$\underset{\substack{\uparrow \\ \text{strain in } y_1-y_2 \text{ axes}}}{\varepsilon_{\alpha\beta}} = l_{\alpha\tilde{\sigma}}\, l_{\beta\tilde{\gamma}}\, \underset{\substack{\uparrow \\ \text{strain in } \tilde{y}_1-\tilde{y}_2 \text{ axes}}}{\tilde{\varepsilon}_{\sigma\gamma}} \qquad \text{⑤}$$

where $\theta = -50°$. The matrix form can be written as:

↑ negative sign appears because we are going clockwise.

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \tilde{\varepsilon}_{11} \\ \tilde{\varepsilon}_{22} \\ \tilde{\varepsilon}_{12} \end{Bmatrix} \quad — ⑥$$

Plugging the strains from (b) and $\theta = -50°$ into equation ⑥,

we get:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} 0.4132 & 0.5868 & -0.9848 \\ 0.5868 & 0.4132 & 0.9848 \\ 0.4924 & -0.4924 & -0.1736 \end{bmatrix} \begin{Bmatrix} 348 \times 10^{-6} \\ 11820 \times 10^{-6} \\ -8210 \times 10^{-6} \end{Bmatrix} \quad — ⑦$$

$$\Rightarrow \boxed{\begin{aligned} \varepsilon_{11} &= 15{,}165 \times 10^{-6} \\ \varepsilon_{22} &= -2997 \times 10^{-6} \\ \varepsilon_{12} &= -4224 \times 10^{-6} \end{aligned}}$$