Practice Problems

6.

$$\begin{Bmatrix} H_1 \\ H_2 \\ H_3 \end{Bmatrix} = \begin{bmatrix} C_{111} & C_{122} & 2C_{112} \\ C_{211} & C_{222} & 2C_{212} \\ C_{311} & C_{322} & 2C_{312} \end{bmatrix} \begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix}$$

- The H subscript must be a free index becouse it changes with equation. It must also be Latin because it takes values 1, 2, 3.
- The M subscript must be Greek because they take values of 1, 2
- The second and third C subscripts are the same as the M subscripts (therefore, they must also be Greek)
- The first C subscript matches the H subscript (therefore, Latin).

If we make the assumptions that $M_{\alpha\beta}$ is symmetric and $C_{i\alpha\beta}$ is symmetric, then the matrix equation given can be written as

$$\boxed{H_i = C_{i\alpha\beta} M_{\alpha\beta}}$$

7.

$$\frac{\partial^2 \varepsilon_{nk}}{\partial y_m \partial y_l} + \frac{\partial^2 \varepsilon_{ml}}{\partial y_n \partial y_k} - \frac{\partial^2 \varepsilon_{nl}}{\partial y_m \partial y_k} - \frac{\partial^2 \varepsilon_{mk}}{\partial y_n \partial y_l} = 0$$

All of the 4 indices are free indices and take on values 1, 2, 3. Therefore, there are 81 ($3 \times 3 \times 3 \times 3$) equations that one could obtain. However, they are not all independent, and in fact, some are trivial. Let's consider the symmetries first.

- $\varepsilon_{nk} = \varepsilon_{kn}$ ⇒ if we reverse the subscripts, we don't get a new equation.
- $\varepsilon_{ml} = \varepsilon_{lm}$
- $\varepsilon_{nl} = \varepsilon_{ln}$
- $\varepsilon_{mk} = \varepsilon_{km}$

The trivial cases are as follows.

$$\left.\begin{array}{l} \circ\ k = l \\ \circ\ m = n \end{array}\right\} \text{ results in } 0 = 0$$

* Note: From the trivial cases, it can be inferred that if

any of the three subscripts have the same value, the equation will be trivial.

These conditions leave only 6 independent equations. Any of the six is acceptable.

<u>$n=k=1,\ m=l=2$</u>

$$\frac{\partial^2 \varepsilon_{11}}{\partial y_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial y_1^2} - \frac{\partial^2 \varepsilon_{12}}{\partial y_1 \partial y_2} - \frac{\partial^2 \varepsilon_{12}}{\partial y_1 \partial y_2} = 0$$

$$\Rightarrow \quad \frac{\partial^2 \varepsilon_{11}}{\partial y_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial y_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial y_1 \partial y_2} = 0 \quad \text{—— (1)}$$

<u>$n=k=1,\ m=2,\ l=3$</u>

$$\frac{\partial^2 \varepsilon_{11}}{\partial y_2 \partial y_3} + \frac{\partial^2 \varepsilon_{23}}{\partial y_1^2} - \frac{\partial^2 \varepsilon_{13}}{\partial y_1 \partial y_2} - \frac{\partial^2 \varepsilon_{12}}{\partial y_1 \partial y_3} = 0 \quad \text{—— (2)}$$

<u>$n=k=1,\ m=l=3$</u>

$$\frac{\partial^2 \varepsilon_{11}}{\partial y_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial y_1^2} - \frac{\partial^2 \varepsilon_{13}}{\partial y_1 \partial y_3} - \frac{\partial^2 \varepsilon_{13}}{\partial y_1 \partial y_3} = 0$$

$$\Rightarrow \quad \frac{\partial^2 \varepsilon_{11}}{\partial y_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial y_1^2} - 2\frac{\partial^2 \varepsilon_{13}}{\partial y_1 \partial y_3} = 0 \quad \text{—— (3)}$$

$n=1,\ k=2,\ m=2,\ l=3$

$$\frac{\partial^2 \varepsilon_{12}}{\partial y_2 \partial y_3} + \frac{\partial^2 \varepsilon_{23}}{\partial y_1 \partial y_2} - \frac{\partial^2 \varepsilon_{13}}{\partial y_2^2} - \frac{\partial^2 \varepsilon_{22}}{\partial y_1 \partial y_3} = 0 \quad \text{—— (4)}$$

$n=1,\ k=2,\ m=l=3$

$$\frac{\partial^2 \varepsilon_{12}}{\partial y_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial y_1 \partial y_2} - \frac{\partial^2 \varepsilon_{13}}{\partial y_2 \partial y_3} - \frac{\partial^2 \varepsilon_{23}}{\partial y_1 \partial y_3} = 0 \quad \text{—— (5)}$$

$n=2,\ k=2,\ m=3,\ l=3$

$$\frac{\partial^2 \varepsilon_{22}}{\partial y_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial y_2^2} - \frac{\partial^2 \varepsilon_{23}}{\partial y_2 \partial y_3} - \frac{\partial^2 \varepsilon_{23}}{\partial y_2 \partial y_3} = 0$$

$$\Rightarrow \quad \frac{\partial^2 \varepsilon_{22}}{\partial y_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial y_2^2} - 2\frac{\partial^2 \varepsilon_{23}}{\partial y_2 \partial y_3} = 0 \quad \text{—— (6)}$$

* Note: Due to the symmetries, other combination of indices $n$, $k$, $m$, $l$ will also give these equations.

To convert to engineering notation, note the following differences.

① Index conversion

| | |
|---|---|
| $11 \rightarrow x$ | $12, 21 \rightarrow xy, yx$ |
| $22 \rightarrow y$ | $13, 31 \rightarrow xz, zx$ |
| $33 \rightarrow z$ | $23, 32 \rightarrow yz, zy$ |

② Shear strain conversion

$$2\varepsilon_{mn} = \gamma_{mn} \quad \text{for} \quad m \neq n$$

③ Axis conversion

$$y_1 \rightarrow x, \quad y_2 \rightarrow y, \quad y_3 \rightarrow z$$

The six equations can be written in engineering notation as follows.

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \quad \text{—— (1)}$$

$$\frac{\partial^2 \varepsilon_x}{\partial y \partial z} + 2\frac{\partial^2 \gamma_{yz}}{\partial x^2} - 2\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} - 2\frac{\partial^2 \gamma_{xy}}{\partial x \partial z} = 0 \quad \text{—— (2)}$$

$$\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} - \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} = 0 \quad \text{—— (3)}$$

$$2\frac{\partial^2 \gamma_{xy}}{\partial y \partial z} + 2\frac{\partial^2 \gamma_{yz}}{\partial x \partial y} - 2\frac{\partial^2 \gamma_{xz}}{\partial y^2} - \frac{\partial^2 \varepsilon_y}{\partial x \partial z} = 0 \quad \text{—— (4)}$$

$$2\frac{\partial^2 \gamma_{xy}}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y \partial x} - 2\frac{\partial^2 \gamma_{xz}}{\partial y \partial z} - 2\frac{\partial^2 \gamma_{yz}}{\partial x \partial z} = 0 \quad \text{—— (5)}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} - \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = 0 \quad \text{—— (6)}$$

These equations are often written in the following form.

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}$$

$$\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2}$$

$$\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} = \frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2}$$

$$2 \frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left( - \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right)$$

$$2 \frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left( \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right)$$

$$2 \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left( \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right)$$