Integral Thicknesses

1 Definitions

The details of the velocity profile $u(y)$ at any x location are rarely significant in engineering applications. The most significant quantities are integral thicknesses which describe the mass flux, momentum flux, and kinetic energy flux in the shear layer:

$$\delta^* = \int \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy \qquad \text{displacement thickness}$$

$$\theta = \int \left(1 - \frac{u}{u_e}\right) \frac{\rho u}{\rho_e u_e}\, dy \qquad \text{momentum thickness}$$

$$\theta^* = \int \left(1 - \frac{u^2}{u_e^2}\right) \frac{\rho u}{\rho_e u_e}\, dy \qquad \text{kinetic energy thickness}$$

2 Integral Thickness Interpretation

These thicknesses appear when comparing the mass, momentum, and kinetic energy flows in a shear layer and a corresponding potential flow.

2.1 Mass flow comparison

Figure 1 shows the mass flux passing between the vertical extent $y = 0 \ldots y_e$ for inviscid and viscous flows with the same edge velocity.

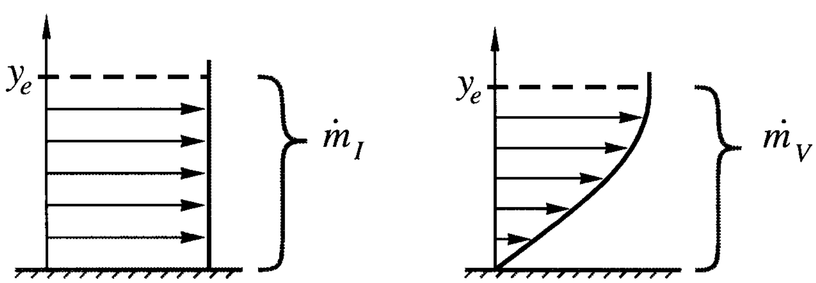


Figure 1: Comparison of inviscid and viscous mass flows

$$\dot{m}_I = \int d\dot{m} = \int_0^{y_e} \rho u\, dy = \dot{m}_I = \rho_e u_e y_e$$

$$\dot{m}_V = \int d\dot{m} = \int_0^{y_e} \rho u\, dy = \rho_e u_e y_e - \int_0^{y_e} (\rho_e u_e - \rho u)\, dy = \dot{m}_I - \rho_e u_e \delta^*$$

The viscous mass flow is decreased by an amount equal to the mass defect $\rho_e u_e \delta^*$.

2.2 Momentum flow comparison

Figure 2 shows the momentum flux carried by the mass flow passing between $y = 0 \ldots y_e$ of the inviscid case. The viscous case capture height is increased by δ^* so that the comparison is done at the same mass flows. In each case, the momentum flow can be considered to be the force acting on a barrier which arrests the flow velocity to zero.

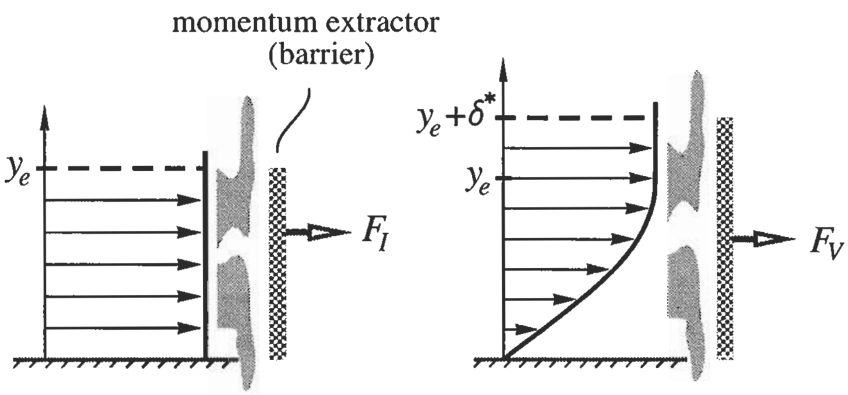


Figure 2: Comparison of inviscid and viscous momentum flows, at the same mass flow

$$F_I = \int u d\dot{m} = \int_0^{y_e} \rho u^2 \, dy = F_I = \rho_e u_e^2 y_e$$

$$F_V = \int u d\dot{m} = \int_0^{y_e+\delta^*} \rho u^2 \, dy = \rho_e u_e^2 y_e - \int_0^{y_e+\delta^*} (u_e - u) \, \rho u dy = F_I - \rho_e u_e^2 \theta$$

The viscous momentum flow is decreased by an amount equal to the momentum defect $\rho_e u_e^2 \theta$.

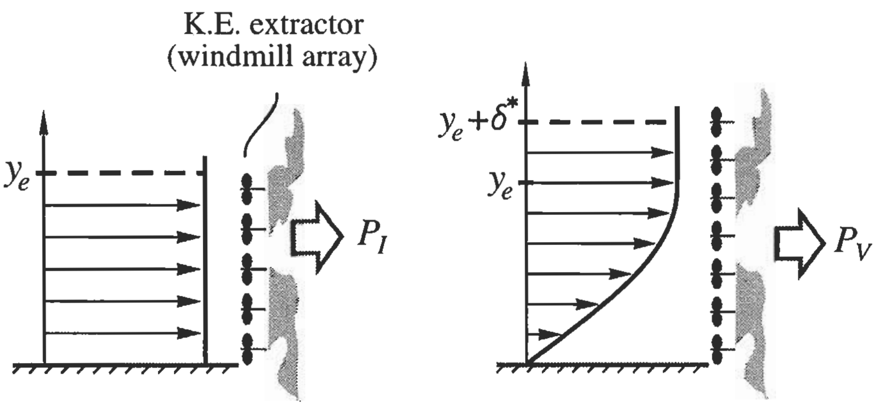
2.3 Kinetic energy flow comparison

Figure 3 shows the kinetic energy flux carried by the mass flow passing between $y = 0 \ldots y_e$ of the inviscid case. The viscous case capture height is again increased by δ^* so that the comparison is done at the same mass flows. In each case, the kinetic energy flow can be considered to be the power generated on an array of perfect windmills which reversibly bring the flow velocity to zero.

$$P_I = \int \frac{1}{2} u^2 d\dot{m} = \int_0^{y_e} \frac{1}{2} \rho u^3 \, dy = P_I = \frac{1}{2} \rho_e u_e^3 y_e$$

$$P_V = \int \frac{1}{2} u^2 d\dot{m} = \int_0^{y_e+\delta^*} \frac{1}{2} \rho u^3 \, dy = \frac{1}{2} \rho_e u_e^3 y_e - \int_0^{y_e+\delta^*} \frac{1}{2} \left(u_e^2 - u^2 \right) \rho u dy = P_I - \frac{1}{2} \rho_e u_e^3 \theta^*$$

The viscous kinetic energy flow is decreased by an amount equal to the kinetic energy defect $\frac{1}{2} \rho_e u_e^3 \theta^*$.



mom

Figure 3: Comparison of inviscid and viscous kinetic energy flows, at the same mass flow