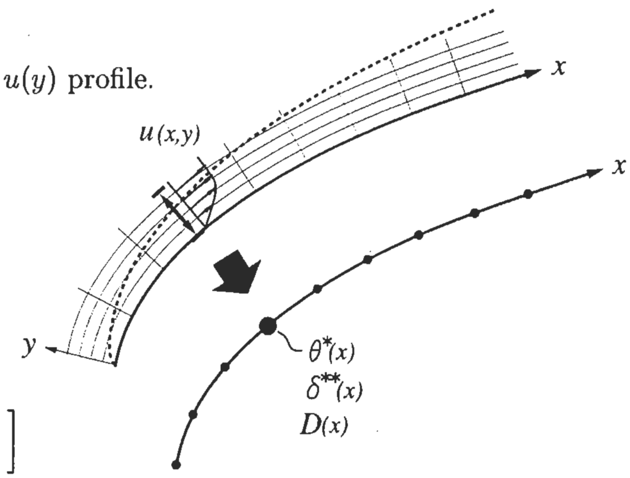
Integral Kinetic Energy Equation

Idea: Integrate BL flow in y, to "wash out" details in $u(y)$ profile.
Converts PDEs in x, y into ODE in x.



Combine Continuity, x-momentum equations:

$$\left(u^2 - u_e^2\right)\left[\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0\right]$$

$$+\ 2u\left[\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = \rho_e u_e\frac{du_e}{dx} + \frac{\partial \tau}{\partial y}\right]$$

$$\Rightarrow\quad 2\rho u^2\frac{\partial u}{\partial x} + (u^2 - u_e^2)\frac{\partial \rho u}{\partial x} + 2\rho u v\frac{\partial u}{\partial y} + (u^2 - u_e^2)\frac{\partial \rho v}{\partial y} = 2u\rho_e u_e\frac{du_e}{dx} + 2u\frac{\partial \tau}{\partial y}$$

or

$$\frac{\partial}{\partial x}\left[(u_e^2 - u^2)\rho u\right] + \frac{\partial}{\partial y}\left[(u_e^2 - u^2)\rho v\right] = -2(\rho_e - \rho)u u_e\frac{du_e}{dx} - 2u\frac{\partial \tau}{\partial y} \qquad (*)$$

Integrate $\int_0^{y_e} (*)\, dy$ term by term:

$$\int_0^{y_e}\frac{\partial}{\partial x}\left[(u_e^2 - u^2)\rho u\right] dy + \int_0^{y_e}\frac{\partial}{\partial y}\left[(u_e^2 - u^2)\rho v\right] dy = -\int_0^{y_e} 2(\rho_e - \rho)u u_e\frac{du_e}{dx}\, dy - \int_0^{y_e} 2u\frac{\partial \tau}{\partial y}\, dy$$

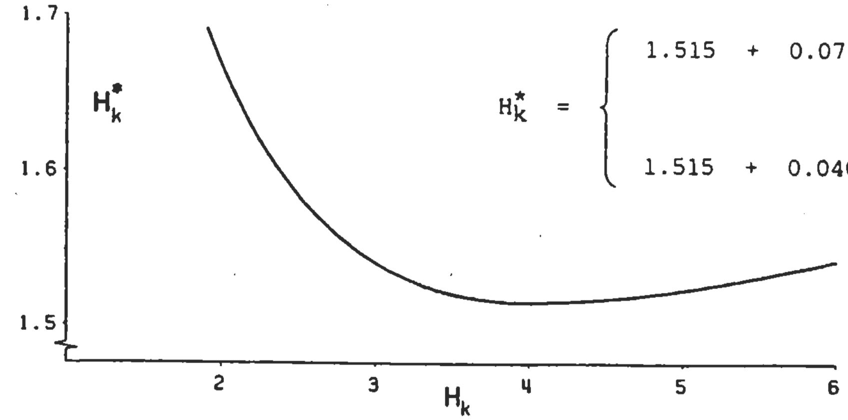
$$\frac{d}{dx}\int_0^{y_e}\left[(u_e^2 - u^2)\rho u\right] dy + 0 = -2u_e\frac{du_e}{dx}\int_0^{y_e}(\rho_e - \rho)u\, dy + 2\int_0^{y_e}\tau\frac{\partial u}{\partial y}\, dy$$

$$\boxed{\frac{d}{dx}\left(\rho_e u_e^3 \theta^*\right) + 2\rho_e u_e^2 \delta^{**}\frac{du_e}{dx} = 2D} \qquad \text{Dimensional form}$$

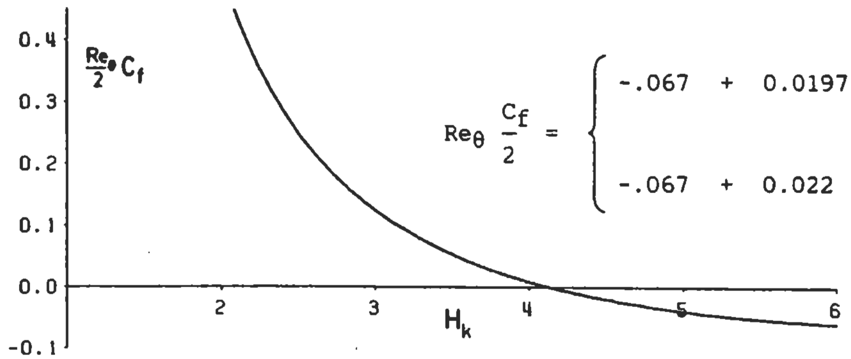
$$\boxed{\frac{d\theta^*}{dx} + \left(\frac{2H^{**}}{H^*} + 3 - M_e^2\right)\frac{\theta^*}{u_e}\frac{du_e}{dx} = 2C_D} \qquad \text{Dimensionless form}$$

Definitions

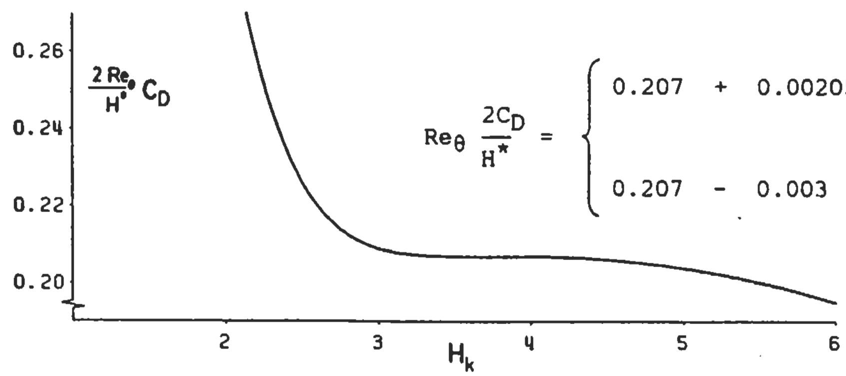
$\theta^* = \int \left(1 - \frac{u^2}{u_e^2}\right)\frac{\rho u}{\rho_e u_e}\, dy$	kinetic energy thickness
$\delta^{**} = \int \left(1 - \frac{\rho}{\rho_e}\right)\frac{u}{u_e}\, dy$	volume flux thickness
$D = \int \tau\frac{\partial u}{\partial y}\, dy$	dissipation integral
$H^* = \frac{\theta^*}{\theta}$	kinetic energy shape parameter
$H^{**} = \frac{\delta^{**}}{\theta}$	density thickness shape parameter
$C_D = \frac{D}{\rho_e u_e^3}$	dissipation coefficient



$$H_k^* = \begin{cases} 1.515 + 0.076\,\dfrac{(H_k-4)^2}{H_k}, & H_k < 4 \\ 1.515 + 0.040\,\dfrac{(H_k-4)^2}{H_k}, & H_k > 4 \end{cases}$$

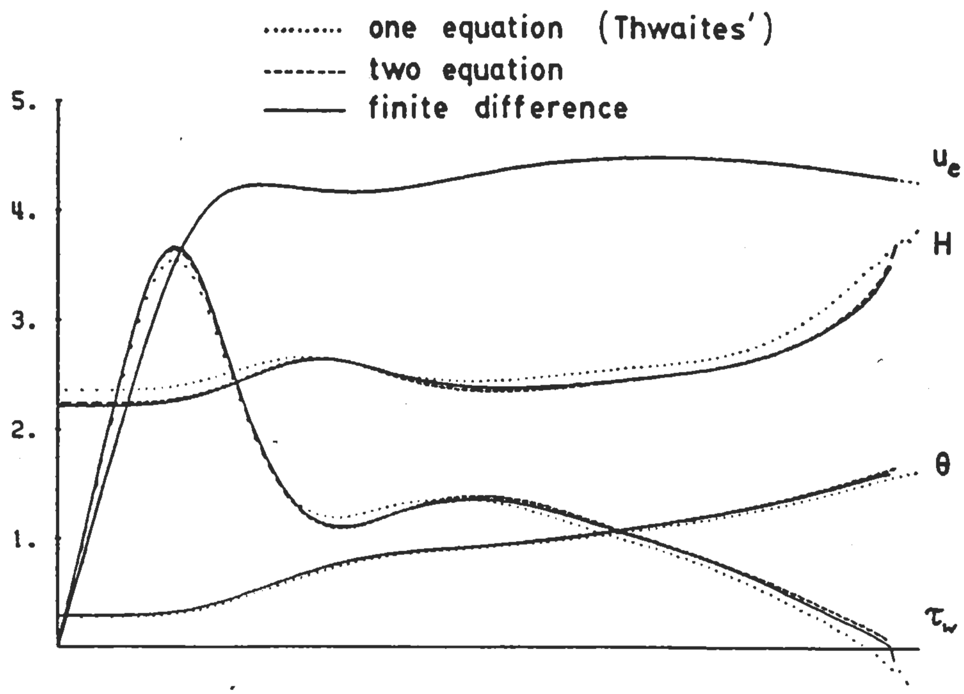


$$Re_\theta \frac{C_f}{2} = \begin{cases} -.067 + 0.01977\,\dfrac{(7.4-H_k)^2}{H_k-1}, & H_k < 7.4 \\ -.067 + 0.022\left(1 - \dfrac{1.4}{H_k-6}\right)^2, & H_k > 7.4 \end{cases}$$



$$Re_\theta \frac{2C_D}{H^*} = \begin{cases} 0.207 + 0.00205\,(4-H_k)^{5.5}, & H_k < 4 \\ 0.207 - 0.003\,(H_k-4)^2, & H_k > 4 \end{cases}$$

Closure relations — two-equation, laminar, integral method.



Comparison — various BL calculation methods