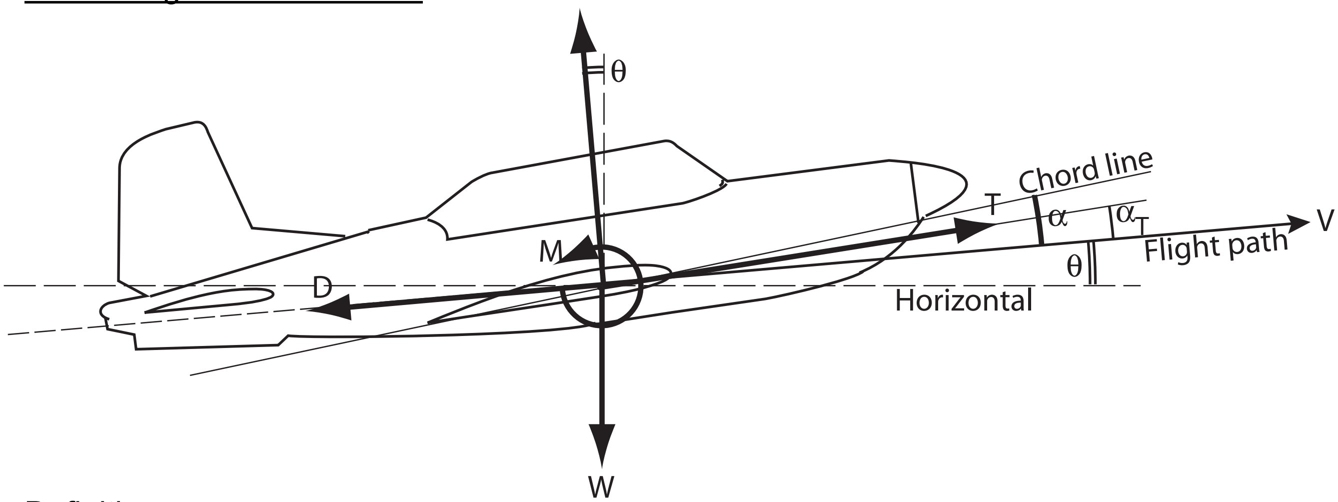
Equations of Aircraft Motion

Force Diagram Conventions



Definitions

$V \equiv$ flight speed

$\theta \equiv$ angle between horizontal & flight path

$\alpha \equiv$ angle of attack (angle between flight path and chord line)

$W \equiv$ aircraft weight

$L \equiv$ lift, force normal to flight path generated by air acting on aircraft

$D \equiv$ drag, force along flight path generated by air acting on aircraft

$M \equiv$ pitching moment

$T \equiv$ propulsive force supplied by aircraft engine/propeller

$\alpha_T \equiv$ angle between thrust and flight path

To derive the equations of motion, we apply

$$\sum \vec{F} = m\vec{a} \qquad (1)$$

Note: we will not be including the potential for a yaw force.

Applying (1) in flight path direction:

$$\sum F_{\parallel} = ma_{\parallel} = m\frac{dV}{dt}$$

and examining the force diagram

$$\sum F_{\parallel} = T\cos\alpha_T - D - W\sin\theta$$

$$\Rightarrow \boxed{T\cos\alpha_T - D - W\sin\theta = m\frac{dV}{dt}} \qquad (2)$$

Now applying (1) in $\perp -$direction to flight path

$$\sum F_{\perp} = ma_{\perp} = m\frac{V^2}{r_c}$$

where $r_c \equiv$ radius of curvature of flight path

$$\sum F_{\perp} = L + T\sin\alpha_T - W\cos\theta$$

$$\Rightarrow \boxed{L + T\sin\alpha_T - W\cos\theta = m\frac{V^2}{r_c}} \qquad (3)$$

Equations (2) & (3) give the equations of motion for an aircraft (neglecting yawing motions) and are quite general. One important specific case of these equations is level, steady flight with the thrust aligned w/ the flight path.

$$\Rightarrow \frac{dV}{dt} = 0, r_c \to \infty, \alpha_T = 0, \theta = 0$$

$$\Rightarrow \boxed{\begin{array}{ll} T = D & \\ & \text{Level, steady flight} \\ L = W & \end{array}}$$

<u>Moment definitions</u>

The pitching moment must be defined relative to a specific location. The two typical locations are:

- leading edge
- $\frac{1}{4}\bar{c}$, quarter of mean chord

Force & Moment Coefficients

Typically, aerodynamicists use non-dimensional force & Moment coefficients.

$$\left.\begin{aligned} C_L &\equiv \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 S} \\ C_D &\equiv \frac{D}{\frac{1}{2}\rho_\infty V_\infty^2 S} \end{aligned}\right\} \text{3D Drag/Lift coefficients}$$

where

ρ_∞ is freestream density

V_∞ is freestream velocity (flight speed)

S is a reference area (problem dependent)

$$\left. q_\infty \equiv \frac{1}{2}\rho_\infty V_\infty^2 \right\} \text{Freestream dynamic pressure}$$

The moment coefficient requires another length scale:

$$C_M \equiv \frac{M}{\frac{1}{2}\rho_\infty V_\infty^2 S \ell_{ref}}$$

$\ell_{ref} \equiv$ reference length scale (problem dependent)

For 2-D problems, such as an airfoil, the forces are actually forces/length. So, for example

3D force	2D force/length
L	L′
D	D′

Similarly, $M \to M'$. The non-dimensional coefficients for 2-D are defined:

$$C_l \equiv \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^2 c_{ref}}$$

$$C_d \equiv \frac{D'}{\frac{1}{2}\rho_\infty V_\infty^2 c_{ref}}$$

$$C_m \equiv \frac{M'}{\frac{1}{2}\rho_\infty V_\infty^2 c_{ref}^2}$$

where c_{ref} is a reference length such as the chord of an airfoil.

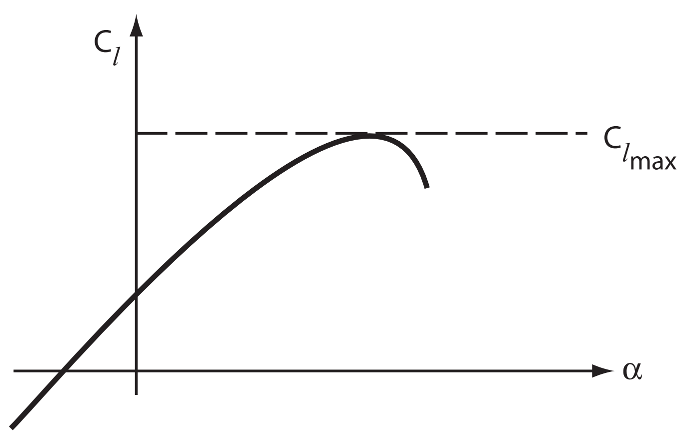
Forces on Airfoils

The forces & moments on airfoils are normalized by the chord length. So,

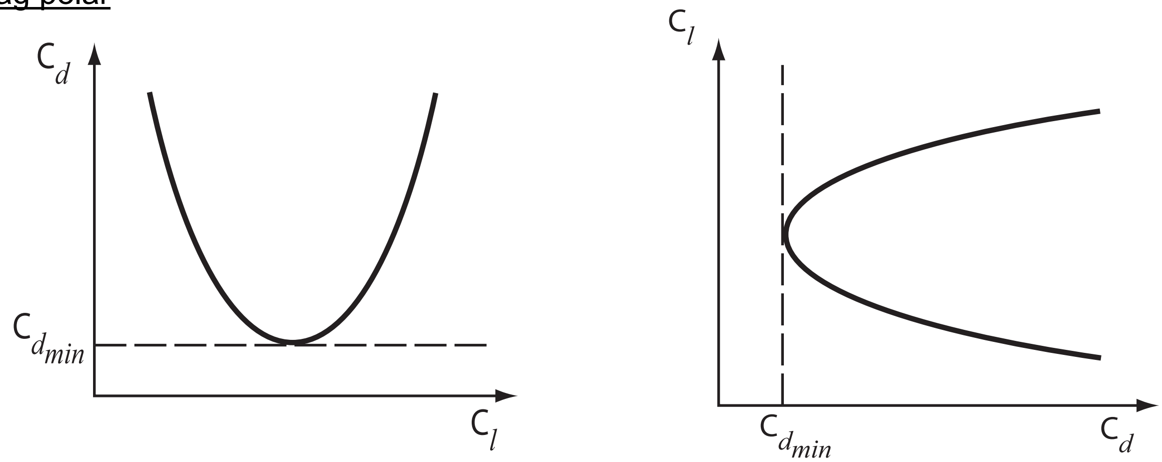
$$C_l \equiv \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^2 c}, \; C_d \equiv \frac{D'}{\frac{1}{2}\rho_\infty V_\infty^2 c}, C_m \equiv \frac{M'}{\frac{1}{2}\rho_\infty V_\infty^2 c^2}$$

Force coefficients data is generally plotted in 2 forms:

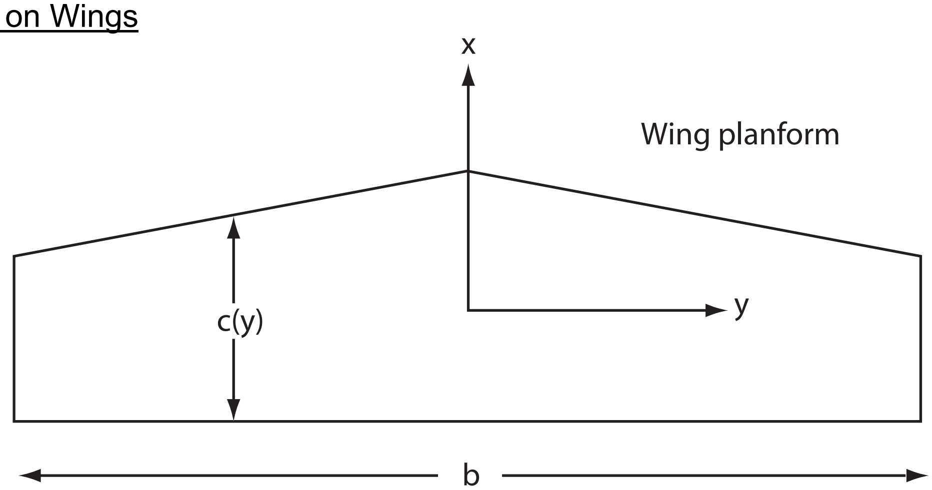
Lift curve



Drag polar



Forces on Wings



$c(y) \equiv$ chord distribution

$b \equiv$ wing span

$S \equiv$ planform area = $\int_{-\frac{b}{2}}^{\frac{b}{2}} c dy$

$A \equiv$ aspect ratio $\equiv \frac{b^2}{S}$

We can think of the 3-D or total lift on the wing as being the sum (i.e. integral) of the 2-D lift acting on the wing.

$\Rightarrow \quad L = \int_{-\frac{b}{2}}^{\frac{b}{2}} L'(y) dy$ where $L'(y)$ = lift distribution

The average 2-D lift on the wing $\bar{L}'$ can be defined:

$$\bar{L}' \equiv \frac{1}{b}\int_{\frac{-b}{2}}^{\frac{b}{2}} L'dy = \frac{L}{b}$$

Plugging that into C_L:

$$C_L \equiv \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 S} = \frac{\int_{-\frac{b}{2}}^{\frac{b}{2}} L'dy}{\frac{1}{2}\rho_\infty V_\infty^2 S} = \frac{\bar{L}'b}{\frac{1}{2}\rho_\infty V_\infty^2 S}$$

But, the average chord or mean chord can be defined as:

$$\bar{c} \equiv \frac{1}{b}\int_{\frac{-b}{2}}^{\frac{b}{2}} cdy = \frac{S}{b}$$

$$\Rightarrow C_L \equiv \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 S} = \frac{\bar{L}'}{\frac{1}{2}\rho_\infty V_\infty^2 \bar{c}}$$

In other words, we can think of the 3-D lift coefficient as the mean value of the 2-D lift coefficient on the wing. The same is true for drag and moment:

$$C_D \equiv \frac{D}{\frac{1}{2}\rho_\infty V_\infty^2 S} = \frac{\bar{D}'}{\frac{1}{2}\rho_\infty V_\infty^2 \bar{c}}$$

$$C_M \equiv \frac{M}{\frac{1}{2}\rho_\infty V_\infty^2 S \ell_{ref}} = \frac{\bar{M}'}{\frac{1}{2}\rho_\infty V_\infty^2 \bar{c}^2}$$

where $\ell_{ref} = \bar{c}$ is used.

<u>A Closer Look at Drag</u>

The drag coefficient can be broken into 2 parts:

$$C_D = \underbrace{C_{D,e}}_{\substack{parasite\\drag}} + \underbrace{\frac{C_L^2}{\Pi e A}}_{\substack{induced\\drag}}$$

where $e =$ span efficiency factor (more on this when we get to lifting line).

The parasite drag contains everything except for induced drag including:

- skin friction drag
- wave drag
- pressure drag (due to separation)

It is a function of α, thus, we can also think of $C_{D,e}$ as being a function of C_L. The parasitic drag can be well-approximated by:

$$C_{D,e} = C_{D_0} + rC_L^2$$

where $C_{D_0} \equiv$ drag at $C_L = 0$, $r =$ empirically determined constant.

$$\Rightarrow C_D = C_{D_0} + \left(r + \frac{1}{\Pi e A} \right) C_L^2$$

Finally, we can re-define e to include r:

$$\Rightarrow \; C_D = C_{D_0} + \frac{1}{\Pi e A} C_L^2$$

where $e \rightarrow \dfrac{e}{1 + r\Pi e A}$.

This re-defined e is known as the Oswald efficiency factor.

We will refer to C_{D_0} as the parasite drag coefficient from now on.