Compressible Viscous Equations

Also known as the compressible Navier-Stokes equations:

Mass: $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho v_j\right) = 0$$

Momentum: $$\frac{\partial\left(\rho v_i\right)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho v_i v_j\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\tau_{ij}\right), \ i = 1,2,3$$

Energy: $$\frac{\partial}{\partial t}(\rho e + \frac{1}{2}\rho v^2) + \frac{\partial}{\partial x_j}\left[(\rho e + \frac{1}{2}\rho v^2) v_j\right] = -\frac{\partial}{\partial x_j}\left(p v_j\right) + \frac{\partial}{\partial x_j}\left(\tau_{ij} v_i\right) + \frac{\partial}{\partial x_j}(\dot{q}_j)$$

$$\tau_{ij} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \delta_{ij}\lambda\frac{\partial v_k}{\partial x_k}$$

$$\dot{q} = k\frac{\partial \mathrm{T}}{\partial x_j}, \qquad e = e\left(p, \mathrm{T}\right) \leftarrow \textit{state relationship}\,(\textit{ideal gas})$$

Incompressible Viscous Equations

In this case, we assume $\rho = const.$

Mass: $$\frac{\partial v_j}{\partial x_j} = 0$$

Momentum: $$\rho\frac{\partial v_i}{\partial t} + \rho\frac{\partial}{\partial x_j}\left(v_i v_j\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right]$$

Usually, $\mu = \mu(\mathrm{T})$. Often, when temperature variations are small, μ =const. is assumed.

$$\Rightarrow \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right] = \mu\left[\frac{\partial v_i}{\partial x_j \partial x_j} + \underbrace{\frac{\partial v_j}{\partial x_j \partial x_i}}_{\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right)=0}\right]$$

$$= \mu\frac{\partial v_i}{\partial x_j \partial x_j}$$

Usual form of momentum for incompressible flow:

$$\rho \frac{\partial v_i}{\partial t} + \rho \frac{\partial}{\partial x_j}(v_i v_j) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial v_i}{\partial x_j \partial x_j}$$

In this case, the energy equation is not needed to find v_i & p.

Incompressible Inviscid Equations

In this case we assume that the effects of viscous stresses are small compared to acceleration and pressure forces:

Mass: $$\frac{\partial v_j}{\partial x_j} = 0$$

Momentum: $$\rho \frac{\partial v_i}{\partial t} + \rho \frac{\partial}{\partial x_j}\left(v_i v_j\right) = -\frac{\partial p}{\partial x_i}$$

These are known as the incompressible Euler equations.

Incompressible Potential Flow

In potential flow, we assume the flow is irrotational (i.e. $\nabla \times \vec{V} = 0$). This allows the velocity to be written as the gradient of a scalar potential:

$$v_i = \frac{\partial \phi}{\partial x_i}, \quad \phi = \text{Potential}$$

Mass: $$\frac{\partial \phi_j}{\partial x_j \partial x_j} = 0$$

Also written out as: $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

or

$$\nabla^2 \phi = 0, \quad \nabla^2 \equiv Laplacian$$

This is a single equation for a single unknown ϕ. It is the same for steady and unsteady flows.

What happens to momentum?