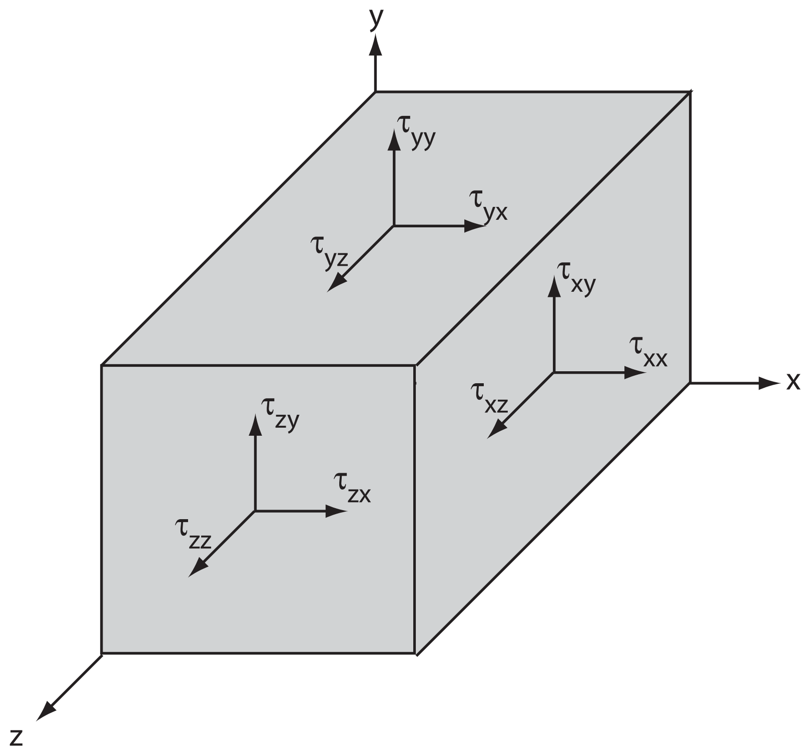
Stress-Strain Relationship for a Newtonian Fluid

First, the notation for the viscous stresses are:

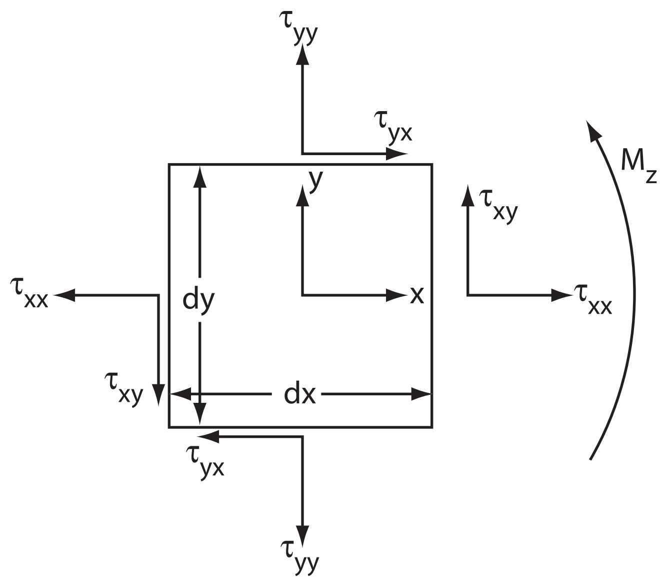


$\tau_{ij} =$ stress acting on the fluid element with a face whose normal is in $+x_i$ direction and the stress is in $+x_j$ direction.

Common assumption is that the net moment created by the viscous stresses are zero.

$\Rightarrow \tau_{ij} = \tau_{ji}$

Let's look at this in 2-D:



M_z = Net moment about center= $\left(\tau_{xy}\frac{dx}{2}\right)dy-\left(\tau_{yx}\frac{dy}{2}\right)dx$
$+\left(\tau_{xy}\frac{dx}{2}\right)dy-\left(\tau_{yx}\frac{dy}{2}\right)dx$

$$\Rightarrow M_z = (\tau_{xy} - \tau_{yx})\frac{dxdy}{2}$$

Thus, for $M_z = 0,\ \tau_{xy} = \tau_{yx}$

Assumptions for Newtonian fluid stress-strain:

1) τ_{ij} is at most a linear function of ε_{ij}.
2) The fluid is isotropic, thus its properties are independent of direction $\Rightarrow$ stress-strain relationship cannot depend on choice of coordinate axes.
3) When the strain rates are zero, the viscous stresses must be zero.

To complete the derivation, we consider the stress-strain relationship in the principal strain axes (i.e. where $\varepsilon_{ij} = 0 \; for \; i \neq j$).

Thus,

$$
\begin{aligned}
\tau_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} \\
\tau_{22} &= C_{21}\varepsilon_{11} + C_{22}\varepsilon_{22} + C_{23}\varepsilon_{33} \\
\tau_{33} &= C_{31}\varepsilon_{11} + C_{32}\varepsilon_{22} + C_{33}\varepsilon_{33}
\end{aligned}
$$

But, to maintain an isotropic relationship:

$$
\begin{aligned}
&C_{11} = C_{22} = C_{33} \\
&C_{12} = C_{21} = C_{31} = C_{13} = C_{23} = C_{32}
\end{aligned}
$$

which leaves only two unknown coefficients.

We define these two coefficients by:

$$\tau_{ii} = 2\mu\varepsilon_{ii} + \lambda(\underbrace{\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}}_{\nabla \bullet \vec{u}})$$

$\mu \equiv$ dynamic viscosity coefficient

$\lambda \equiv$ 2^{nd} or bulk viscosity coefficient

For general axes (i.e. not in principal axes):

$$
\begin{aligned}
&\tau_{ij} = 2\mu\varepsilon_{ij} + \delta_{ij}\lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \\
&\text{or, } \tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda\nabla \bullet \vec{u}
\end{aligned}
$$

where

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$