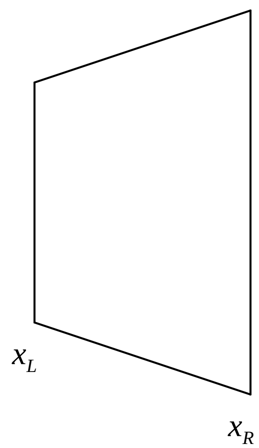
Behavior of Isentropic Flow in Quasi-1D

Recall cons. of mass:
$\rho u A = const.$

Consider a perturbation in the area

$$\rho_L = \rho \qquad p_L = p \qquad u_L = u \qquad A_L = A$$



$$\rho_R = \rho_L + d\rho \qquad p_R = p_L + dp \qquad u_R = u_L + du \qquad A_R = A_L + dA$$

$$\rho u A = (\rho + d\rho)(u + du)(A + dA)$$
$$= \rho u A + d\rho u A + \rho A du + \rho u dA + H.O.T.$$
$$\Rightarrow u A d\rho + \rho A du + \rho u dA = 0$$

Or

$$\boxed{\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0} \qquad (1)$$

Similar manipulations to x-momentum gives:

$$\boxed{dp + \rho u du = 0} \qquad (2)$$

Also, since the flow is assumed isentropic, we can use the definition of the speed of sound to relate $d\rho$ and dp:

$$a^2 = \left.\frac{\partial p}{\partial \rho}\right|_{s=const.}$$

$$\Rightarrow \boxed{dp = a^2 d\rho}$$ Since we have assumed $s = const.$ in the flow (3)

We are interested in how the flow properties change when the area changes. So, we use (2) and (3) to eliminate terms from (1). For example, let's determine how u changes with A:

$$\frac{d\rho}{\rho}+\frac{du}{u}+\frac{dA}{A}=0$$

$$\frac{dp}{\rho a^2}+\frac{du}{u}+\frac{dA}{A}=0$$

$$-\frac{\rho u du}{\rho a^2}+\frac{du}{u}+\frac{dA}{A}=0$$

$$(1-M^2)\frac{du}{u}+\frac{dA}{A}=0$$

$$\Rightarrow \quad \boxed{\frac{du}{u}=-\frac{1}{1-M^2}\frac{dA}{A}}$$

<u>When $M<1$</u>: $*\frac{du}{u}>0$ when $\frac{dA}{A}<0$

$\Rightarrow u$ increases when A decreases

$*\frac{du}{u}<0$ when $\frac{dA}{A}>0$

$\Rightarrow u$ decreases when A increases

This is what we expect from our understanding of incompressible flow.

<u>When $M>1$</u>: $*\frac{du}{u}>0$ when $\frac{dA}{A}>0$

$\Rightarrow u$ increases when A increases!

$*\frac{du}{u}<0$ when $\frac{dA}{A}<0$

$\Rightarrow u$ decreases when A decreases!

This is very different from incompressible flow.

What's happening for $M>1$? Clearly, ρu must behave the opposite of A regardless of the Mach number. So, what must be happening is that ρ changes more rapidly than A for $M>1$ and, thus u behaves opposite of what we expect from subsonic flow behavior. Let's check this:

$$\frac{du}{u}=-\frac{1}{1-M^2}\frac{dA}{A}$$

$$-\frac{dp}{\rho u^2}=-\frac{1}{1-M^2}\frac{dA}{A}$$

$$\boxed{\frac{dp}{\rho u^2}=\frac{1}{1-M^2}\frac{dA}{A}}$$

$$\frac{a^2 d\rho}{\rho u^2} = \frac{1}{1-M^2}\frac{dA}{A}$$

$$\boxed{\frac{d\rho}{\rho} = \frac{M^2}{1-M^2}\frac{dA}{A}}$$

These results show that, regardless of whether $M > 1$ or $M < 1$, dp and $d\rho$ have the same sign as dA:

$$\boxed{\begin{array}{l} p,\rho \uparrow \text{ when } A\uparrow \\ p,\rho \downarrow \text{ when } A\downarrow \end{array}}$$

Recall conservation of mass:

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

$$\frac{M^2}{1-M^2}dA - \frac{1}{1-M^2}\frac{dA}{A} + \frac{dA}{A} = 0$$

$$\boxed{\frac{dp}{\rho u^2} = \frac{1}{1-M^2}\frac{dA}{A}}$$

$$\frac{a^2 d\rho}{\rho u^2} = \frac{1}{1-M^2}\frac{dA}{A}$$

$$\boxed{\frac{d\rho}{\rho} = \frac{M^2}{1-M^2}\frac{dA}{A}}$$

Thus we find that:

$M < 1$: $p,\rho \uparrow$ when $A\uparrow$

$p,\rho \downarrow$ when $A\downarrow$

$M > 1$: $p,\rho \uparrow$ when $A\downarrow$

$p,\rho \downarrow$ when $A\uparrow$

Which are the opposite of how u behaves. It is also useful to consider the magnitudes of the different changes:

$$\frac{d\rho}{\rho}+\frac{du}{u}+\frac{dA}{A}=0$$

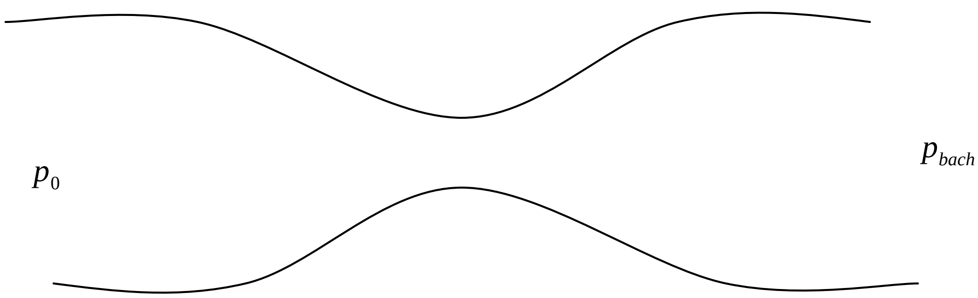
$$\frac{M^2}{1-M^2}\frac{dA}{A}-\frac{1}{1-M^2}\frac{dA}{A}+\frac{dA}{A}=0$$

Flow low M, ρ changes are small compared to u changes. But for $M>1$, ρ changes more rapidly.

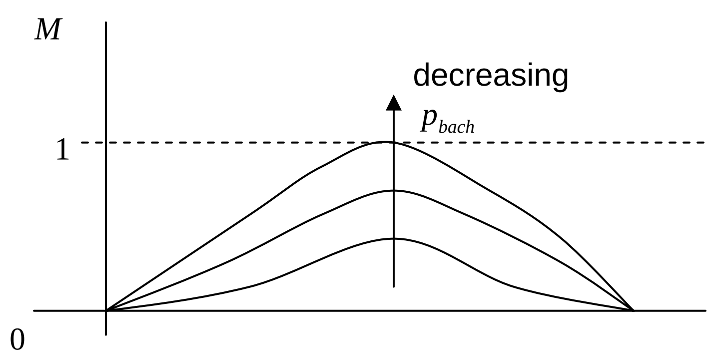
$*$ Also, of interest is that $d\rho, dp,$ and du become extremely large as $M \to 1$.

$*$ In fact, this requires that $M=1$ must occur at a minimum in the area where $dA=0$.

Let's consider a converging-diverging duct:



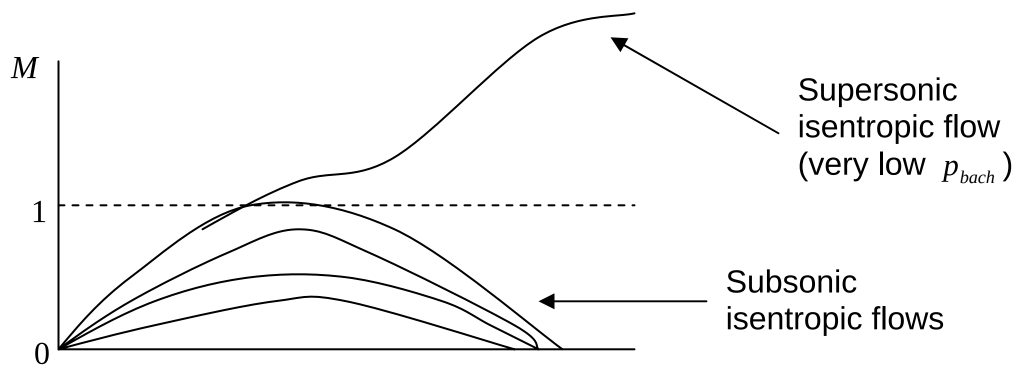
When p_{bach} is only a little less than p_0, the flow will be subsonic:



$*$ As p_{bach} is lowered from p_o, we will eventually hit the p_{bach} at which $M=1$ at the throat.

What happens if p_{bach} is lowered further?

There is one other isentropic flow through this geometry which occurs when p_{bach} is very low. In this situation, the flow will become supersonic in the divergent section:


M
1
0
Supersonic
isentropic flow
(very low p_{bach})
Subsonic
isentropic flows