Solutions of the Laminar Boundary Layer Equations

The boundary layer equations for incompressible steady flow, i.e.,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp_e}{dx} + \mu \frac{\partial^2 u}{\partial y^2}$$

Note: since $\frac{\partial p}{\partial y} = 0$, we set $p = p_e(x)$, i.e. the boundary layer edge pressure.

have been solved for a handful of important cases. We will look at the results for a flat plate and a family of solutions called Falkner-Skan Solutions.

<u>Flat Plate (Laminar): Blasius Solution</u>

For a flat plate, $p_e = p_\infty \leftarrow$ constant

$$\Rightarrow \frac{dp_e}{dx} = 0$$

Blasius was able to show that the boundary later equations could be rewritten to only depend on a parameter,

$$\eta \equiv y\sqrt{\frac{V_\infty}{2\nu x}}$$

and its derivatives

The resulting solution has been tabulated and compared to experiments on the following page. Note:

$$u(x,y) = V_\infty f'(\eta) \qquad \text{where } f' = \frac{df}{d\eta}$$

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{\mu V_\infty f''(0)}{\sqrt{2\nu x / V_\infty}}$$

These values from the solution of $f(\eta)$ can be used to find:

$$\delta_{99\%} \equiv y - \text{location at which } u(x,y) = 0.99V_\infty$$

From the table, $f'(\eta) = 0.99$ at $\eta \approx 3.5$:

$$\eta = y\sqrt{\frac{V_\infty}{2\nu x}}$$

$$3.5 = \delta_{99\%}\sqrt{\frac{V_\infty}{2\nu x}}$$

$$\Rightarrow \delta_{99\%} = 3.5\sqrt{\frac{2\nu x}{V_\infty}} \quad \leftarrow \text{boundary layer grows as } \sqrt{x}$$

Typically, this result is written "non dimensionally" as:

$$\frac{\delta_{99\%}}{x} = \frac{5.0}{\sqrt{\mathrm{Re}_x}} \text{ where } \underbrace{\mathrm{Re}_x}_{\text{Reynold's number based on } x} \equiv \frac{V_\infty x}{\nu}$$

We can also find:

$$\frac{\delta^*}{x} = \frac{1.7208}{\sqrt{\mathrm{Re}_x}}$$

$$\frac{\theta}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

Comment:

* At leading edge of a flat plate $x \to 0$ and this gives $C_f \to \infty$!
* In reality, the leading edge of an infinitely thin plate would have very large, but not infinite C_f.
* The problem is that near the leading edge of a thin plate, the boundary layer equations are not correct and the Navier-Stokes equations are needed. Question: Why did the boundary layer approximation fail at $x \to 0$?