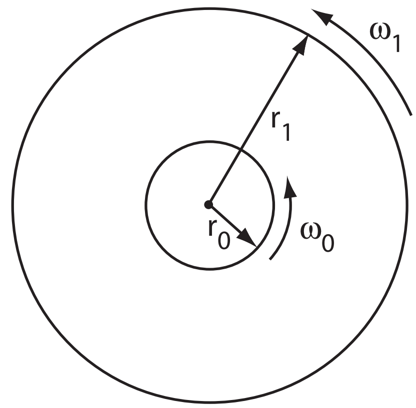
Problem #1

Assume:

- Incompressible
- 2-D flow $\Rightarrow V_z = 0, \frac{\partial}{\partial z} = 0$
- Steady $\Rightarrow \frac{\partial}{\partial t} = 0$
- Parallel $\Rightarrow V_r = 0$



a) Conservation of mass for a 2-D flow is:

$$\frac{1}{r}\frac{\partial}{\partial r}(r \underbrace{V_r}_{=0}) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta) = 0$$

$$\Rightarrow \frac{\partial}{\partial \theta}(V_\theta) = 0 \Rightarrow V_\theta \text{ does not depend on } \theta$$

$$\Rightarrow \boxed{V_\theta = V_\theta(r)}$$

b) θ-mometum equation is:

$$\underbrace{\frac{\partial V_\theta}{\partial t}}_{steady} + (\vec{V}\cdot\nabla)V_\theta + \underbrace{\frac{V_r V_\theta}{r}}_{V_r=0} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu(\nabla^2 V_\theta + \frac{2}{r^2}\underbrace{\frac{\partial V_r}{\partial \theta}}_{V_r=0} - \frac{V_\theta}{r^2})$$

In cylindrical coordinates:

$$(\vec{V}\cdot\nabla) = \underbrace{V_r}_{=0}\frac{\partial}{\partial r} + \frac{1}{r}V_\theta\frac{\partial}{\partial \theta}$$

Thus,

$$(\vec{V}\cdot\nabla)V_\theta + \frac{1}{r}V_\theta \underbrace{\frac{\partial V_\theta}{\partial \theta}}_{\substack{=0 \text{ from} \\ \text{continuity}}} = 0$$

Also,

$$\nabla^2 V_\theta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2}}_{=0}$$

Combining all of these results gives:

$$\frac{1}{\rho r}\frac{\partial p}{\partial \theta} = \underbrace{\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta}{\partial r}\right) - \frac{V_\theta}{r^2}\right]}_{\text{this side is independent of } \theta}$$

Since the right-hand-side (RHS) is independent of θ, this requires that $\frac{\partial p}{\partial \theta} = \text{constant}$ for fixed r. But as θ varies from $0 \to 2\pi$, it must be equal at $0 \,\&\, 2\pi$, that is $p(\theta = 0) = p(\theta = 2\pi)$. If not, the solution would be discontinuous. Thus, $\frac{\partial p}{\partial \theta} = 0 \Leftarrow$ constant must be zero!

The differential equation for V_θ is:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V_\theta}{\partial r}) - \frac{V_\theta}{r^2} = 0$$

A little rearranging gives:

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rV_\theta)\right] = 0$$

Integrating once gives:

$$\frac{1}{r}\frac{d}{dr}(rV_\theta) = C_1$$

Integrating again gives:

$$rV_\theta = \frac{1}{2}C_1 r^2 + C_2$$

$$\Rightarrow \boxed{V_\theta = \frac{1}{2}C_1 r + \frac{C_2}{r}}$$

Next, we must apply the no-slip boundary conditions to find V_θ. Specifically,

at $r = r_o, V_\theta = \omega_o r_o$

at $r = r_1, V_\theta = \omega_1 r_1$

because flow velocity equals wall velocity in a viscous flow.

So, apply $r = r_o$ & $r = r_1$ bc's:

$$\left.\begin{aligned} \omega_o r_o &= \frac{1}{2}C_1 r_o + \frac{C_2}{r_o} \\ \omega_1 r_1 &= \frac{1}{2}C_1 r_1 + \frac{C_2}{r_1} \end{aligned}\right\} \Rightarrow \left\{ \begin{aligned} \frac{1}{2}C_1 &= \frac{\omega_1 \frac{r_1}{r_o} - \omega_o \frac{r_o}{r_1}}{\frac{r_1}{r_0} - \frac{r_0}{r_1}} \\ &\Rightarrow \\ C_2 &= \frac{r_o r_1(\omega_o - \omega_1)}{\frac{r_1}{r_0} - \frac{r_0}{r_1}} \end{aligned}\right.$$

Or, rearranged a little gives:

$$\boxed{V_\theta = r_o\omega_o \frac{r_1/r - r/r_1}{r_1/r_o - r_o/r_1} + r_1\omega_1 \frac{r/r_o - r_o/r}{r_1/r_o - r_o/r_1}}$$

c) The radial momentum equation is:

$$\frac{\partial V_r}{\partial t} + (\vec{V} \cdot \nabla)V_r - \frac{1}{r}V_\theta^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 V_r - \frac{V_r}{r^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta}\right)$$

But $V_r = 0$ & $\frac{\partial V_\theta}{\partial \theta} = 0$ so this reduces to:

$$\frac{\partial p}{\partial r} = \frac{\rho V_\theta^2}{r}$$

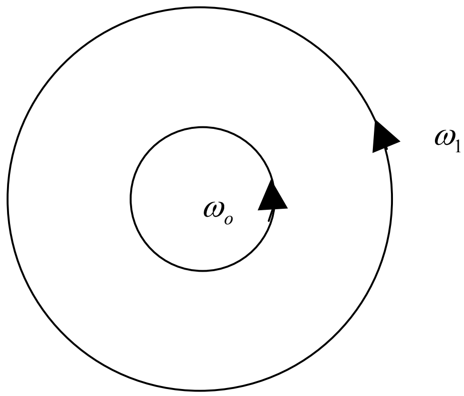
Since $\frac{\rho V_\theta^2}{r} \geq 0$ always, then clearly $\frac{\partial p}{\partial r} \geq 0$.

Thus, pressure increases with r.

d) On the inner cylinder, the moment is a result of the skin friction due to the fluid shear stress. For this flow in which only $V_\theta \neq 0$ and is only a function of r, the only non-zero shear stress is $\tau_{r\theta}$ and has the following form:

$$\tau_{r\theta} = \mu \underbrace{\left(\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r} \right)}_{=\varepsilon_{r\theta}\text{, the only non-zero strain}} = \mu \left[r \frac{\partial}{\partial r} \left(\frac{V_\theta}{r} \right) \right]$$

Rotating Cylinders



For the problem you studied in the homework:

1. What direction is the fluid element acceleration?

2. What direction are the net pressure forces on a fluid element?

3. What direction are the net viscous forces on a fluid element?