Force Calculations for Lifting Line

Recall:

$$\Gamma(y) = \Gamma(\theta) = 2bV_\infty \sum_{n=1}^{N} A_n \sin n\theta$$

$$y = -\frac{b}{2}\cos\theta$$

The local two-dimensional lift distribution is given by Kutta-Joukowsky:

$$L'(y) = \rho V_\infty \Gamma(y)$$

$$\Rightarrow \boxed{L'(\theta) = 2b\rho V_\infty^2 \sum_{n=1}^{N} A_n \sin n\theta}$$

To calculate the total wing lift, we integrate L':

$$L = \int_{-\frac{b}{2}}^{\frac{b}{2}} L'(y)dy \qquad dy = \frac{b}{2}\sin\theta d\theta$$

$$= \int_{o}^{\pi} \left[2b\rho V_\infty^2 \sum_{n=1}^{N} A_n \sin n\theta \right] \left(\frac{b}{2}\sin\theta d\theta \right)$$

But: $\displaystyle\int_{o}^{\pi} \sin m\theta \sin k\theta d\theta = \begin{cases} 0, m \neq k \\ \frac{\pi}{2}, m = k \end{cases}$

In this case, $m = n$ and $k = 1$. So, the only non-zero term is for $n = 1$.

$$\Rightarrow \qquad L = (2b\rho V_\infty^2)\left(A_n \frac{\pi}{2} \right)\left(\frac{b}{2} \right)$$

$$\Rightarrow \qquad L = \frac{\pi}{2} b^2 \rho V_\infty^2 A_1$$

$$\Rightarrow \qquad \boxed{C_L = \frac{L}{\frac{1}{2}\rho V_\infty^2 S} = \frac{\pi b^2 A_1}{S} = \pi \text{A} A_1}$$

The induced drag is similar. In this case:

$$D_i' = \rho V_\infty \alpha_i(y)\Gamma(y)$$

From previous lecture,

$$\alpha_i(\theta) = \sum_{n=1}^{N} nA_n \frac{\sin n\theta}{\sin\theta}$$

$$\Rightarrow \qquad D_i' = \rho V_\infty \left(\sum_{n=1}^{N} nA_n \frac{\sin n\theta}{\sin\theta}\right)\left(2bV_\infty \sum_{m=1}^{N} A_n \sin m\theta\right)$$

Integrating along the wing:

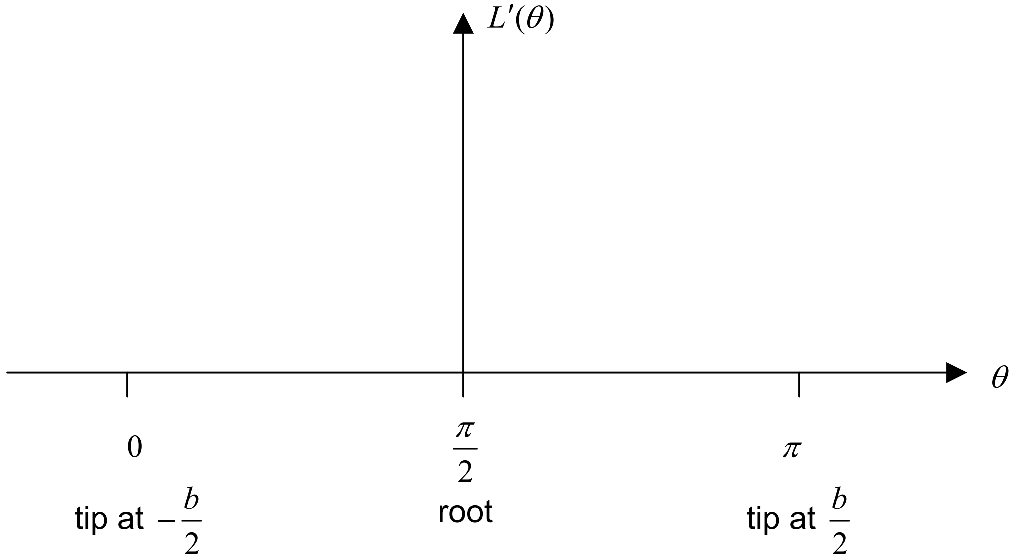
$$\begin{aligned} D_i &= \int_{-\frac{b}{2}}^{\frac{b}{2}} D_i'(y)dy \\ &= b^2\rho V_\infty^2 \int_0^\pi \left(\sum_{n=1}^{N} nA_n \frac{\sin n\theta}{\sin\theta}\right)\left(\sum_{m=1}^{N} A_m \sin m\theta\right)(\sin\theta d\theta) \\ &= b^2\rho V_\infty^2 \underbrace{\int_0^\pi \left(\sum_{n=1}^{N} nA_n \sin n\theta\right)\left(\sum_{m=1}^{N} A_m \sin m\theta\right)d\theta}_{\text{only} \neq 0 \text{ for } n=m} \\ &= b^2\rho V_\infty^2 \sum_{n=1}^{N} nA_n^2 \frac{\pi}{2} \end{aligned}$$

$$D_i = \frac{\pi}{2} b^2 \rho V_\infty^2 \sum nA_n^2$$

$$\boxed{\begin{aligned} &C_{D_i} = \frac{D_i}{\frac{1}{2}\rho V_\infty^2 S} = \pi \mathrm{A} \sum_{n=1}^{N} nA_n^2 \\ &\text{or } \; C_{D_i} = \frac{C_L^2}{\pi \mathrm{A}}(1+\delta), \\ &\text{where } \; \delta = \sum_{n=2}^{N} n\left(\frac{A_n}{A_1}\right)^2 \end{aligned}}$$

Lift Distributions

The lift distributions due to each of the A_n terms can be plotted as well:



Elliptic Lift Distribution

Recall that minimum induced drag is achieved when $A_n = 0$ for $n > 1$. In this case:

$$L'(\theta) = 2b\rho V_\infty^2 \sum A_n \sin n\theta$$
$$L'(\theta) = 2b\rho V_\infty^2 A_1 \sin\theta$$

but: $$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{y}{b/2}\right)^2}$$

$$\Rightarrow \quad \boxed{L'(y) = 2b\rho V_\infty^2 A_1 \sqrt{1-\left(\frac{y}{b/2}\right)^2}} \quad \text{Elliptic lift}$$