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16.07 Dynamics
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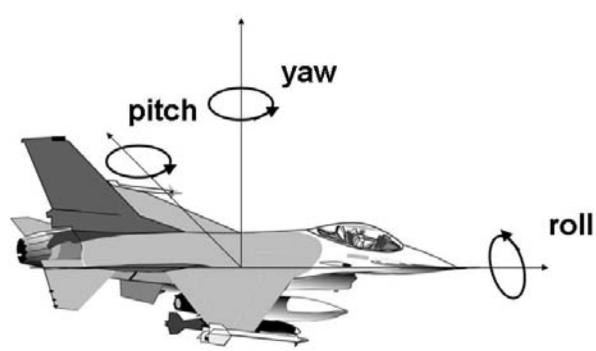
Lecture L29 - 3D Rigid Body Dynamics

3D Rigid Body Dynamics: Euler Angles

The difficulty of describing the positions of the body-fixed axis of a rotating body is approached through the use of Euler angles: spin $\dot{\psi}$, nutation θ and precession ϕ shown below in Figure 1. In this case we surmount the difficulty of keeping track of the principal axes fixed to the body by making their orientation the **unknowns** in our equations of motion; then the angular velocities and angular accelerations which appear in Euler's equations are expressed in terms of these fundamental unknowns, the positions of the principal axes expressed as angular deviations from some initial positions.

Euler angles are particularly useful to describe the motion of a body that rotates about a fixed point, such as a gyroscope or a top or a body that rotates about its center of mass, such as an aircraft or spacecraft. Unfortunately, there is no standard formulation nor standard notation for Euler angles. We choose to follow one typically used in physics textbooks. However, for aircraft and spacecraft motion a slightly different one is used; the primary difference is in the definition of the "pitch" angle. For aircraft motion, we usually refer the motion to a horizontal rather than to a vertical axis. In a description of aircraft motion, ψ would be the "roll" angle; ϕ the "yaw" angle; and θ the "pitch" angle. The pitch angle θ would be measured from the horizontal rather than from the vertical, as is customary and useful to describe a spinning top.



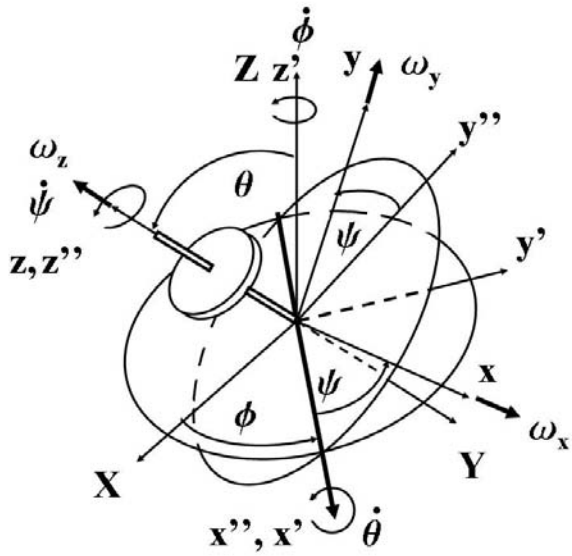
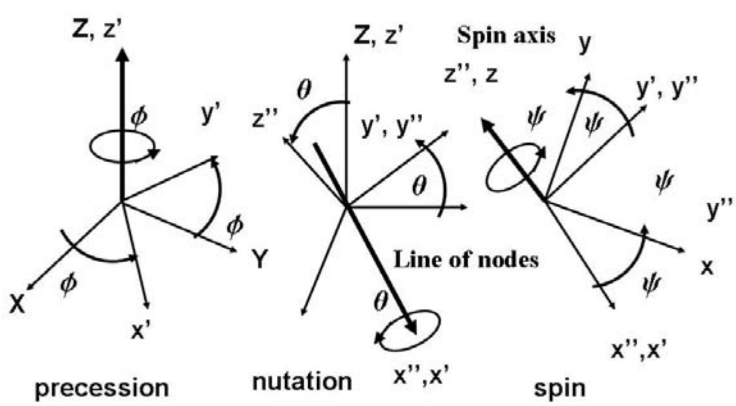


Figure 1: Euler Angles

In order to describe the angular orientation and angular velocity of a rotating body, we need three angles. As shown on the figure, we need to specify the rotation of the body about its "spin" or z body-fixed axis, the angle ψ as shown. This axis can also "precess" through an angle ϕ and "nutate" through an angle θ.
To develop the description of this motion, we use a series of transformations of coordinates, as we did in Lecture 3. The final result is shown below. This is the coordinate system used for the description of motion of a general three-dimensional rigid body described in body-fixed axis.
To identify the new positions of the principal axes as a result of angular displacement through the three Euler angles, we go through a series of coordinate rotations, as introduced in Lecture 3.



We first rotate from an initial X, Y, Z system into an x', y', z' system through a rotation ϕ about the Z, z' axis. The angle ϕ is called the angle of precession.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cos\phi & sin\phi & 0 \\ -sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = [T_1] \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

The resulting x', y' coordinates remain in the X, Y plane. Then, we rotate about the x' axis into the x'', y'', z'' system through an angle θ. The x'' axis remains coincident with the x' axis. The axis of rotation for this transformation is called the "line of nodes". The plane containing the x'', y'' coordinate is now tipped through an angle θ relative to the original X, Y plane. The angle θ is called the angle of nutation.

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = [T_2] \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}.$$

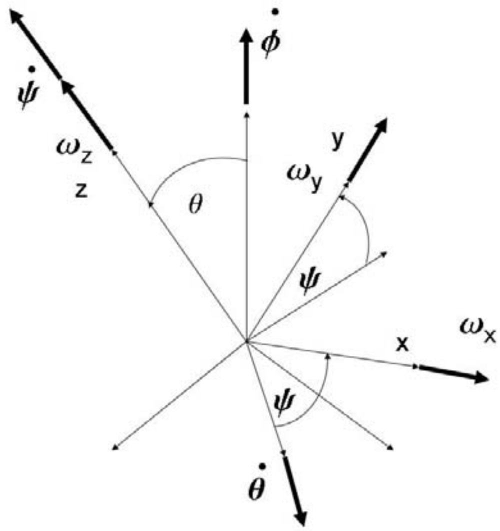
And finally, we rotate about the z'', z system through an angle ψ into the x, y, z system. The z'' axis is called the spin axis. It is coincident with the z axis. The angle ψ is called the spin angle; the angular velocity $\dot{\psi}$ the spin velocity.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = [T_3] \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}.$$

The final "Euler" transformation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = [T_3][T_2][T_1] \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} cos\psi cos\phi - cos\theta sin\phi sin\psi & cos\psi sin\phi + cos\theta cos\phi sin\psi & sin\theta sin\psi \\ -sin\psi cos\phi - cos\theta sin\phi cos\psi & -sin\psi sin\phi + cos\theta cos\phi cos\psi & sin\theta cos\psi \\ sin\phi sin\theta & -cos\phi sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

This is the final x, y, z body-fixed coordinate system for the analysis, with angular velocities $\omega_x, \omega_y, \omega_z$ as shown. The individual coordinate rotations $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ give us the angular velocities. However, these vectors do not form an orthogonal set: $\dot{\phi}$ is along the original Z axis; $\dot{\theta}$ is along the line of nodes or the x' axis; while $\dot{\psi}$ is along the z or spin axis.



This is easily reorganized by taking the components of these angular velocities about the final x, y, z coordinate system using the Euler angles, giving

$$\omega_x = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \tag{1}$$

$$\omega_y = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \tag{2}$$

$$\omega_z = \dot{\phi} \cos\theta + \dot{\psi} \tag{3}$$

We could press on, developing formulae for angular momentum, and changes in angular momentum in this coordinate system, applying these expressions to Euler's equations and develop the complete set of governing differential equations. In general, these equations are very difficult to solve. We will gain more understanding by selecting a few simpler problems that are characteristic of the more general motions of rotating bodies.

3D Rigid Body Dynamics: Free Motions of a Rotating Body

We consider a rotating body in the absence of applied/external moments. There could be an overall gravitational force acting through the center of mass, but that will not affect our ability to study the rotational motion about the center of mass independent of such a force and the resulting acceleration of the center of mass. (Recall that we may equate moments to the rate of change of angular momentum about the center of mass even if the center of mass is accelerating.) Such a body could be a satellite in rotational motion in orbit. The rotational motion about its center of mass as described by the Euler equations will be independent of its orbital motion as defined by Kepler's laws. For this example, we consider that the body is symmetric such that the moments of inertia about two axis are equal, $I_{xx} = I_{yy} = I_0$, and the moment of inertia about z is I. The general form of Euler's equations for a free body (no applied moments) is

$$0 = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z \tag{4}$$

$$0 = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x \tag{5}$$

$$0 = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \tag{6}$$

For the special case of a symmetric body for which $I_{xx} = I_{yy} = I_0$ and $I_{zz} = I$ these equations become

$$0 = I_0\dot{\omega}_x - (I_0 - I)\omega_y\omega_z \tag{7}$$

$$0 = I_0\dot{\omega}_y - (I - I_0)\omega_z\omega_x \tag{8}$$

$$0 = I\dot{\omega}_z \tag{9}$$

We conclude that for a symmetric body, ω_z, the angular velocity about the spin axis, is constant. Inserting this result into the two remaining equations gives

$$I_0\dot{\omega}_x = ((I_0 - I)\omega_z)\omega_y \tag{10}$$

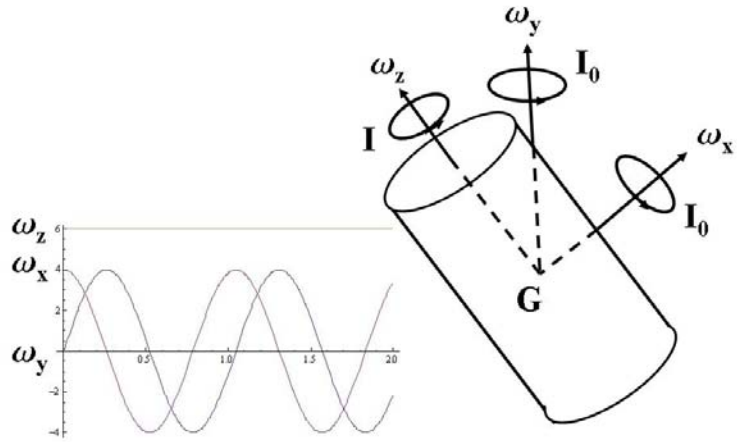
$$I_0\dot{\omega}_y = -((I_0 - I)\omega_z)\omega_x. \tag{11}$$

Since ω_z is **constant**, this gives two linear equations for the unknown ω_x and ω_y. Assuming a solution of the form $\omega_x = A_x e^{i\Omega t}$ and $\omega_y = A_y e^{i\Omega t}$, whereas before we intend to take the real part of the assumed solution, we obtain the following solution for ω_x and ω_y

$$\omega_x = A\cos\Omega t \tag{12}$$

$$\omega_y = A\sin\Omega t \tag{13}$$

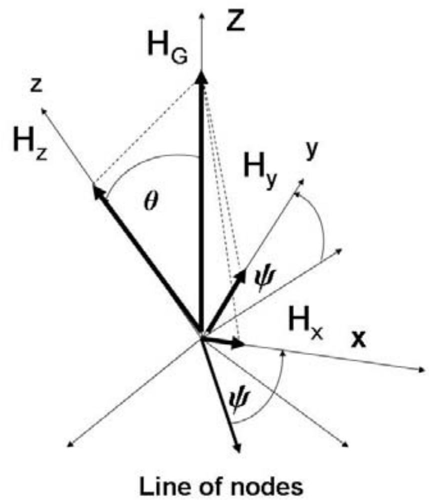
where $\Omega = \omega_z(I - I_0)/I_0$ and A is determined by initial conditions. Since ω_z is constant, the total angular velocity $\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} = \sqrt{A^2 + \omega_z^2}$ is constant. The example demonstrates the direct use of the Euler equations. Although the components of the $\boldsymbol{\omega}$ vector can be found from the solution of a linear equation, additional work must be done to find the actual position of the body. The body motion predicted by this solution is sketched below.



The x, y, z axis are body fixed axis, rotating with the body; the solutions for $\omega_x(t), \omega_y(t)$ and ω_z give the components of $\boldsymbol{\omega}$ following these moving axis. If angular velocity transducers were mounted on the body to measure the components of $\boldsymbol{\omega}$, $\omega_x(t), \omega_y(t)$ and ω_z from the solution to the Euler equations would be obtained, shown in the figure as functions of time. Clearly, as seen from a fixed observer this body undergoes a complex spinning and tumbling motion. We could work out the details of body motion as seen by a fixed observer. (See Marion and Thornton for details.) However, this is most easily accomplished by reformulating the problem expressing Euler's equation using Euler angles.

Description of Free Motions of a Rotating Body Using Euler Angles

The motion of a free body, no matter how complex, proceeds with an angular momentum vector which is constant in direction and magnitude. For body-fixed principle axis, the angular momentum vector is given by $\boldsymbol{H}_G = I_{xx}\omega_x + I_{yy}\omega_y + I_{zz}\omega_z$. It is convenient to align the constant angular momentum vector with the Z axis of the Euler angle system introduced previously and express the angular momentum in the i, j, k system. The angular momentum in the x, y, z system, $\boldsymbol{H}_G = \{H_x, H_y, H_Z\}$ is obtained by applying the "Euler" transformation to the angular momentum vector expressed in the X, Y, Z system, $\boldsymbol{H}_G = \{0, 0, H_G\}$.



$$\boldsymbol{H}_G = H_G \sin\theta \sin\psi \boldsymbol{i} + H_G \sin\theta \cos\psi \boldsymbol{j} + H_G \cos\theta \boldsymbol{k} \tag{14}$$

Then the relationship between the angular velocity components and the Euler angles and their time derivative given in Eq.(1-3) is used to express the angular momentum vector in the Euler angle coordinate system.

$$H_G \sin\theta \sin\psi \quad = I_{xx}\omega_x = \quad I_{xx}(\dot{\phi} sin\theta \sin\psi + \dot{\theta}\cos\psi) \tag{15}$$

$$H_G \sin\theta \cos\psi \quad = I_{yy}\omega_y = \quad I_{yy}(\dot{\phi} sin\theta \cos\psi - \dot{\theta}\sin\psi) \tag{16}$$

$$H_G \cos\theta \quad = I_{zz}\omega_z = \quad I_{zz}(\dot{\phi}\cos\theta + \dot{\psi}) \tag{17}$$

where H_G is the magnitude of the $\boldsymbol{H}_G$ vector.

The first two equation can be added and subtracted to give expressions for $\dot{\phi}$ and $\dot{\theta}$. Then a final form for spin rate $\dot{\psi}$ can be found, resulting in

$$\dot{\phi} \quad = \quad H_G(\frac{\cos^2\psi}{I_{yy}} + \frac{\sin^2\psi}{I_{xx}}) \tag{18}$$

$$\dot{\theta} \quad = \quad H_G(\frac{1}{I_{xx}} - \frac{1}{I_{yy}})\sin\theta \sin\psi \cos\psi \tag{19}$$

$$\dot{\psi} \quad = \quad H_G(\frac{1}{I_{zz}} - \frac{\cos^2\psi}{I_{yy}} - \frac{\sin^2\psi}{I_{xx}})\cos\theta \tag{20}$$

For constant H_G, these equations constitute a first order set of non-linear equations for the Euler angle ϕ, θ and ψ and their time derivatives $\dot{\phi}, \dot{\theta}$ and $\dot{\psi}$. In the general case, these equations must be solved numerically. Considerable simplification and insight can be gained for axisymmetric bodies for which $I_{xx} = I_{yy} = I_0$ and $I_{zz} = I$. In this case, we have

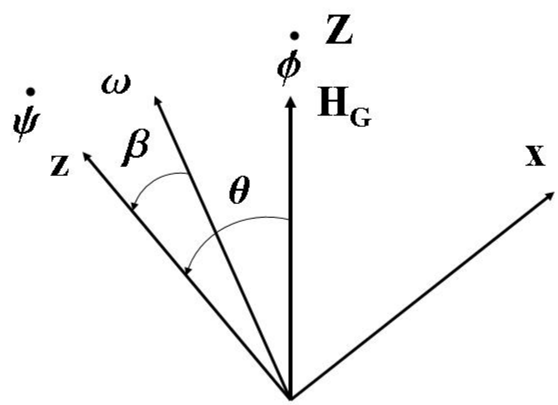
$$\dot{\phi} = H_G/I_0 \tag{21}$$

$$\dot{\theta} = 0 \tag{22}$$

$$\dot{\psi} = H_G(\frac{1}{I} - \frac{1}{I_0})\cos\theta \tag{23}$$

Thus, in this case, the nutation angle θ is constant; the spin velocity $\dot{\psi}$ is constant, and the precession velocity $\dot{\phi}$ is constant. Note that if I_0 is greater that I, $\dot{\phi}$ and $\dot{\psi}$ are of the same sign; and if I_0 is less than I, $\dot{\phi}$ and $\dot{\psi}$ are of opposite signs; for $I_0 = I$, the problem falls apart since we are now dealing with the inertial equivalent of a sphere which will not exhibit precession.

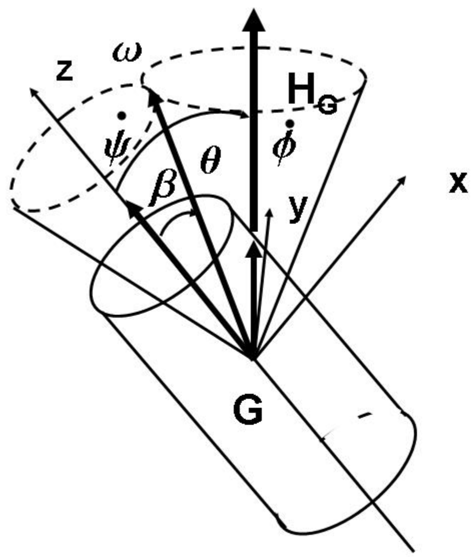
We now examine the geometry of the solution in detail. We assume that the body has some initial angular momentum that could have arisen from an earlier impulsive moment applied to the body or from a set of initial conditions set by an earlier motion. In this case, we would know both the magnitude of the angular momentum H_G and its angle θ from the spin axis. The geometry of the solution is shown below. We align the angular momentum vector with the Z axis.



For general motion of an axisymmetric body, the angular momentum $\boldsymbol{H}_G$ and the angular velocity $\boldsymbol{\omega}$ vectors are not parallel. Using the spin axis z as a reference, the angular momentum $\boldsymbol{H}_G$ makes an angle θ with the spin axis; the angular velocity $\boldsymbol{\omega}$ makes an angle β with the spin axis. For this body, although the angular momentum is not aligned with the angular velocity, they must be in the x, z plane; this is a consequence of symmetry: that $I_{yy} = I_{xx} = I_0$. Therefore both ω_y and H_{Gy} must be zero. We examine the system in the plane formed by $\boldsymbol{H}_g$ and $\boldsymbol{\omega}$, as shown in the figure. The relation between the angular velocities $\dot{\psi}$ and

$\dot{\phi}$ must be such that the vectors $I_0\omega_y\mathbf{j} + I\omega_z\mathbf{k} = \boldsymbol{H}_G$. Since $\boldsymbol{H}_G$ makes an angle θ with the z axis, and $\boldsymbol{\omega}$ makes an angle β with the Z axis, we have

$$tan\theta = H_{Gx}/H_{Gz} = (I_0\omega_x)/(I\omega_z) = (I_0/I)tan\beta. \tag{24}$$



What we would see in this motion is a body spinning about the z axis with $\dot{\psi}$ and "precessing" about the Z axis with $\dot{\phi}$ –this is essentially a definition of precession. The $\boldsymbol{\omega}$ vector, which is the instantaneous axis of rotation, is in the x, z plane.

We now focus on the Z axis, and investigate the motion that can exist for a freely rotating body with the given parameters when it has an angular momentum in the Z direction. We see that a body rotating about its own axis with angular velocity $\dot{\psi}$ at an angle θ from the Z axis, will also undergo a steady precession of $\dot{\phi}$ about the Z axis, keeping the angular momentum $\boldsymbol{H}_G$ directed along the Z axis. The instantaneous axis of rotation maintains a fixed angle β from the spin axis z. The $\boldsymbol{\omega}$ vector maintains a fixed angle $\theta - \beta$ from the vertical. The motion is that of the body cone rolling around the space cone.

A variety of useful and general relations can be written between the various components of angular velocity $\boldsymbol{\omega}$ and angular momentum $\boldsymbol{H}$, the Euler angles ϕ and ψ and the angles θ and β. Using the notation H_G and

ω for the magnitude of the $\boldsymbol{H}_G$ and $\boldsymbol{\omega}$ vectors, we have

$$\omega_x = \omega \sin\beta \tag{25}$$

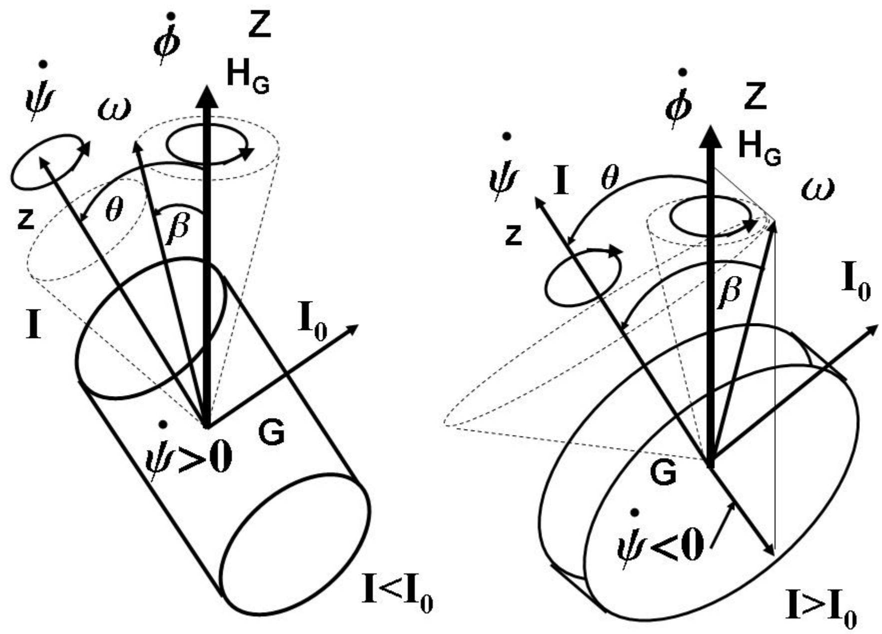
$$\omega_z = \omega \cos\beta \tag{26}$$

$$H_G = \sqrt{H_x^2 + H_z^2} = \sqrt{(\omega_x I_0)^2 + (\omega_z I)^2} = \omega\sqrt{I_0^2 sin^2\beta + I^2 cos^2\beta} \tag{27}$$

$$\dot{\phi} = \omega\sqrt{(sin^2\beta + (I/I_0)^2 cos^2\beta)} \tag{28}$$

$$\dot{\psi} = (1 - (I/I_0))\,\omega cos\beta \tag{29}$$

$$\tag{30}$$



Two different solution geometries are shown: one for which $I_0 > I$; one for which $I_0 < I$. Consider first the case where $I < I_O$. This would be true for a long thin body such as a spinning football, a reentering slender missile or an F-16 in roll.

In this case, the spin $\dot{\psi}$ and the precession $\dot{\phi}$ are of the same sign. The body precesses about the angular momentum vector while spinning. The vector $\boldsymbol{\omega}$ is the instantaneous axis of rotation. The instantaneous axis of rotation is the instantaneous tangent between the body cone and the space cone. The **outside** of the body cone, shown in the figure as a cone of half-angle β aligned along the body axis, rotates about the **outside** of the space cone, shown in the figure as a cone of half-angle $\theta - \beta$ with its axis aligned along the angular momentum vector. This is called direct precession. For direct precession, $\beta < \theta$.

Now consider the case where $I > I_0$; this would be true for a frisbee, a flat spinning satellite, or a silver dollar tossed into the air.

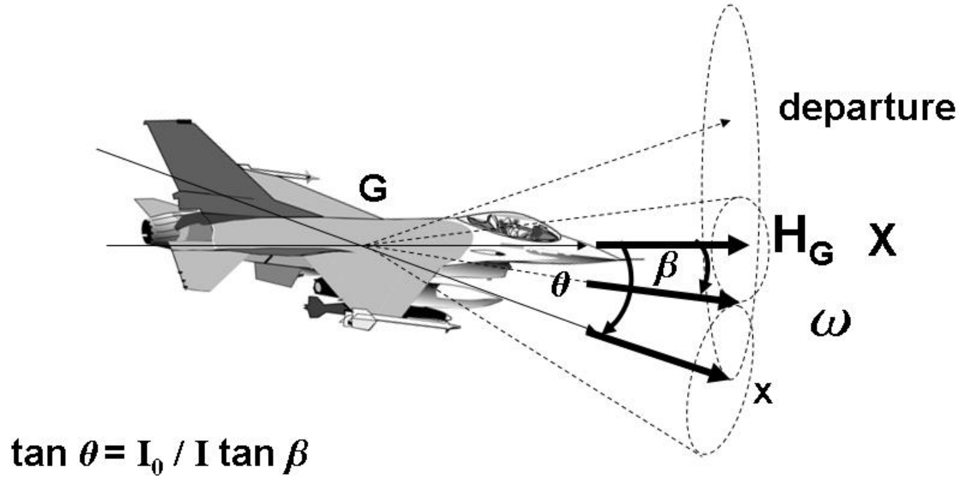
In this case, the spin $\dot{\psi}$ and the precession $\dot{\phi}$ are of opposite signs. This means that the $\boldsymbol{\omega}$ vector is on the other side of the angular momentum vector relative to the spin axis z. The body still precesses about the angular momentum vector while spinning. The vector $\boldsymbol{\omega}$ is still the instantaneous axis of rotation. But now, the **inside** of the body cone rotates about the **outside** of the space cone. The angle β is greater than the angle θ. The body cone shown in the figure as a cone of half-angle β aligned along the body axis; the space cone has a half-angle $\beta - \theta$ and is aligned along the angular momentum vector. This is called retrograde procession; the rotations are in opposite directions. Depending upon body geometry, one of these solutions would be obtained whenever the angular momentum vector is not directed along a single principle axis. This would occur if an impulsive moment is applied to a body along any axis which is not a principle axis. An example of this is discussed below.

Extreme Aircraft Dynamics

The dynamics of aircraft have traditionally been dominated by aerodynamic forces. The location of the center of mass relative to the aerodynamic center was an important consideration as was the question of whether the vertical tail provided enough yaw moment to keep the vehicle flying straight. The details of the response of aerodynamic forces to small disturbances of the vehicle in pitch, roll and yaw, determined the stability of the aircraft and the frequency of the various longitudinal and lateral stability modes.

With the introduction of high-performance fighter aircraft, whose moments of inertia about all three axes were comparable, which had the ability to initiate rapid rolling, and whose roll axis was not a principal axis, we entered into a new flight regime where, at the limit, dynamics dominated aerodynamics.

Consider this limit, where dynamics dominates aerodynamics. We have a F-16 at high altitude where atmospheric density is small, and we have a test pilot giving a strong roll input about the aircraft roll axis which is not a principal axis. For simplicity, we take the y and z inertias to be equal and equal to I_0. Low aspect ratio aircraft have moments of inertia in roll that are less than that in pitch or yaw so that this example is given by the case $I_0 > I$ and we may apply the free-body spinning solution just discussed. We consider that an impulsive moment/torque is applied about the roll axis and inquire what free-body motion this would set up. At this limit, we are neglecting aerodynamics forces. In agreement with our previous analysis, the moment about the x axis would produce an angular momentum about the x axis. But since the x axis is not a principal axis, we would initiate a coning motion as shown.



This could come as quite surprise to a pilot. The question of what happens next is dependent upon the details of the aerodynamic forces, but just to comment on the historical record: in the first days of testing high-performance fighter aircraft, several test pilots lost control of their aircraft, in some cases with fatal results. This phenomenon, called roll coupling, is now well understand and incorporated into the design and testing of new fighter aircraft.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition 7/9

W.T. Thompson, *Introduction to Space Dynamics*, Chapter 5

J. H. Ginsberg, *Advanced Engineering Dynamics*, Second Edition, Chapter 8

J.B Marion and S.T. Thornton, *Classical Dynamics*, Chapter 10

J.C. Slater and N.H. Frank, *Mechanics*, Chapter 6

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