

## Lecture L27 - 3D Rigid Body Dynamics: Kinetic Energy; Instability; Equations of Motion

### 3D Rigid Body Dynamics

In Lecture 25 and 26, we laid the foundation for our study of the three-dimensional dynamics of rigid bodies by: 1.) developing the framework for the description of changes in angular velocity due to a general motion of a three-dimensional rotating body; and 2.) developing the framework for the effects of the distribution of mass of a three-dimensional rotating body on its motion, defining the principal axes of a body, the inertia tensor, and how to change from one reference coordinate system to another.

We now undertake the description of angular momentum, moments and motion of a general three-dimensional rotating body. We approach this very difficult general problem from two points of view.

The first is to prescribe the motion in term of given rotations about fixed axes and solve for the force system required to sustain this motion.

The second is to study the "free" motions of a body in a simple force field such as gravitational force acting through the center of mass or "free" motion such as occurs in a "zero-g" environment. The typical problems in this second category involve gyroscopes and spinning tops. The second set of problems is by far the more difficult.

Before we begin this general approach, we examine a case where kinetic energy can give us considerable insight into the behavior of a rotating body. This example has considerable practical importance and its neglect has been the cause of several system failures.

### Kinetic Energy for Systems of Particles

In Lecture 11, we derived the expression for the kinetic energy of a system of particles. Here, we derive the expression for the kinetic energy of a system of particles that will be used in the following lectures. A typical particle,  $i$ , will have a mass  $m_i$ , an absolute velocity  $\mathbf{v}_i$ , and a kinetic energy  $T_i = (1/2)m_i\mathbf{v}_i \cdot \mathbf{v}_i = (1/2)m_iv_i^2$ . The total kinetic energy of the system,  $T$ , is simply the sum of the kinetic energies for each particle,

$$T = \sum_{i=1}^n T_i = \sum_{i=1}^n \frac{1}{2}m_iv_i^2 = \frac{1}{2}mv_G^2 + \sum_{i=1}^n \frac{1}{2}m_iv_i'^2 , .$$

where  $v'$  is the velocity relative to the center of mass. Thus, we see that the kinetic energy of a system of particles equals the kinetic energy of a particle of mass  $m$  moving with the velocity of the center of mass, plus the kinetic energy due to the motion of the particles relative to the center of mass,  $G$ .

We have said nothing about the conservation of energy for a system of particles. As we shall see, that depends upon the details of internal interactions and the work done by the external forces. The same is true for a rigid body. Work done by internal stresses, or energy lost due to the complexities of a system described as a rigid body, but which in reality may have internal modes which drain energy, will act to decrease the kinetic energy. Thus, although we can confidently relate angular momentum to external forces, we have no such confidence in dealing with conservation of energy. As we shall see, this has important technological implications, especially for the stability of spacecraft. In this section we consider the particles to be components of a rigid body.

## Kinetic Energy for a 3D Rigid Body

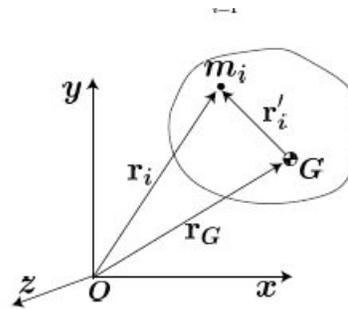
For a rigid body, the summation  $i = 1, n$  becomes an integral over the total mass  $M$ .

$$T = \int_m \frac{1}{2} v^2 dm = \frac{1}{2} M v_G^2 + \int_m \frac{1}{2} v'^2 dm .$$

For a rigid body, the velocity relative to the center of mass is written

$$\vec{v}' = \vec{\omega} \times \vec{r}' \tag{1}$$

where  $\vec{r}'$  is the vector to the mass  $dm$  for the center of mass  $G$ .



Using the vector identity

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{A} \cdot (\vec{B} \times \vec{C}), \tag{2}$$

we have

$$v'^2 = \vec{v}' \cdot \vec{v}' = (\vec{\omega} \times \vec{r}') \cdot (\vec{\omega} \times \vec{r}') = \vec{\omega} \cdot (\vec{r}' \times (\vec{\omega} \times \vec{r}')) = \vec{\omega} \cdot \vec{r}' \times \vec{v}'$$

.

Therefore,

$$\int_m v'^2 dm = \vec{\omega} \cdot \int_m \vec{r}' \times \vec{v}' dm = \vec{\omega} \cdot \vec{H}_G, \tag{3}$$

and the total expression for kinetic energy of a rigid body can be written

$$T = \frac{1}{2}Mv_G^2 + \vec{\omega} \cdot \vec{H}_G = \frac{1}{2}\vec{v}_G \cdot \vec{L}_G + \frac{1}{2}\vec{\omega} \cdot \vec{H}_G \quad (4)$$

where  $M\vec{L}_G$  is the linear momentum.

If there is a fixed point  $O$  about which the body rotates, then

$$T = \frac{1}{2}\vec{\omega} \cdot \vec{H}_0$$

where  $\vec{H}_0$  is the angular momentum about point  $O$ . Even if there is no fixed point about which the body rotates, there is an instantaneous center of rotation  $C$ . In some situations, it may be useful to write

$$T_C = \frac{1}{2}\vec{\omega} \cdot \vec{H}_C \quad (5)$$

If the angular velocity is expressed in principal axes, then the angular momentum about the center of mass can be written  $\vec{H}_G = I_{xx}\omega_x\vec{i} + I_{yy}\omega_y\vec{j} + I_{zz}\omega_z\vec{k}$  so that the kinetic energy can be written

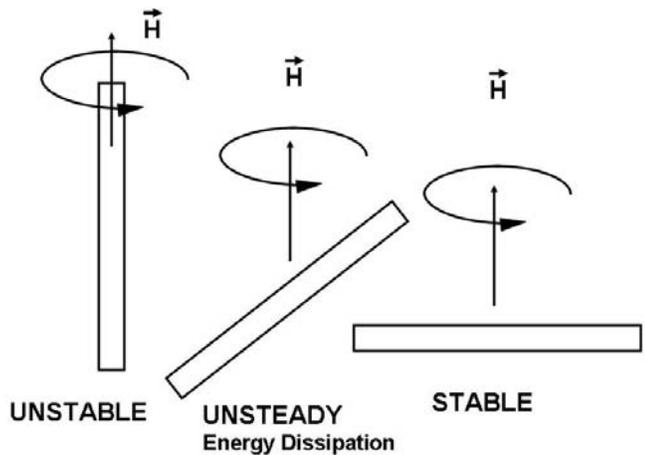
$$T = \frac{1}{2}Mv_G^2 + \frac{1}{2}(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2) \quad (6)$$

The above expressions for kinetic energy are useful to apply the principle of work and energy. We see that the kinetic energy has two components: one is due to the translation,  $\frac{1}{2}Mv_G^2$ . The other is due to rotation. The rotational component can be written as

$$T_R = \frac{1}{2}\vec{\omega} \cdot \vec{H}_G = \frac{1}{2}(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2) \quad (7)$$

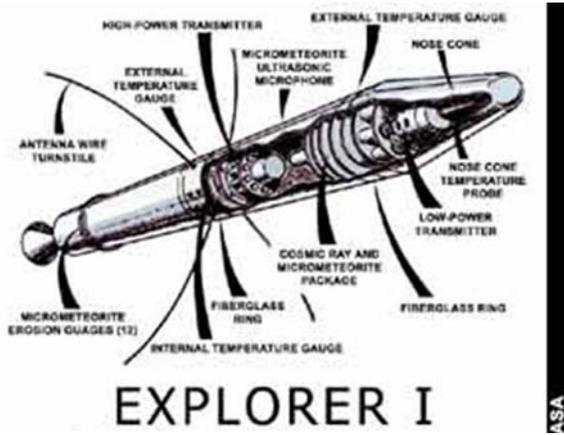
We have no assurance that energy will be conserved. Internal motions can dissipate energy. This is not true for angular momentum: internal forces and motions will not change angular momentum. The implications of this can be profound.

Consider a body spinning about the  $z$  axis of inertia with no external moments. Then the angular momentum will be constant and since it is a vector, it will always point in the  $z$  direction so that  $\vec{H}_G = H_z\vec{k}$  will be constant. Now let us assume that  $I_{zz} < I_{xx}$  and  $I_{zz} < I_{yy}$ . In this case, for a given angular momentum, the kinetic energy is a maximum, since  $T = \frac{1}{2}H_z^2/I_{zz}$ , and  $I_{zz}$  is the smallest moment of inertia. If there is any dissipation mechanism—structural damping, fuel sloshing, friction—then the angular momentum will stay constant in magnitude and direction while the energy decreases. The only way, that the energy can decrease while keeping the angular momentum constant is to change the axis of spin to one with a larger moment of inertia while keeping the vector direction and magnitude of the angular momentum vector constant. Therefore, if the body tumbles and begins to spin about the  $x$  or  $y$  body-fixed axis, the kinetic energy becomes  $T = \frac{1}{2}H_z^2/(I_{xx} \text{ or } I_{yy})$ . The kinetic energy will be reduced since both  $I_{xx}$  and  $I_{yy}$  are greater than  $I_{zz}$ .



The sketch shows the process by which this would occur. An initial spin about the body's  $z$  axis will transition over time to a spin about the body's  $x$  or  $y$  axis, while keeping the  $\vec{H}_G$  constant in magnitude and direction. The initial state is simple; the final state is simple; the intermediate states are complex in that the body axis is not aligned with the rotation vector, much like a spinning top performing a complex motion. Do we have to worry about this?

The first American satellite, Explorer 1, was built by the Jet Propulsion Laboratory in Pasadena, and launched from Canaveral on 31 January 1958 by a modified Jupiter-C missile into an orbit ranging between 354 kilometers and 2,515 kilometers at an angle of 33 degrees to the equator. The satellite was long and slender. It was spin-stabilized about its long axis—the one with the smallest moment of inertia. Spin stabilization has advantages in that the angular momentum remains fixed in direction in space under passive control, if it's stable! The satellite had flexible antenna which vibrated and dissipated energy. During the mission, it unexpectedly suffered an attitude failure in that it tipped over and began to spin about the axis with the maximum moment of inertia. Other space systems as well as re-entry vehicles are prone to this phenomena.



# Governing Equations for Rotational Motion of a Three-Dimensional Body

The governing equations are those of conservation of linear momentum  $\mathbf{L} = M\mathbf{v}_G$  and angular momentum,  $\mathbf{H} = [I]\boldsymbol{\omega}$ , where we have written the moment of inertia in matrix form to remind us that in general the direction of the angular momentum is not in the direction of the rotation vector  $\boldsymbol{\omega}$ . Conservation of linear momentum requires

$$\dot{\mathbf{L}} = \mathbf{F} \quad (8)$$

Conservation of angular momentum, about a fixed point  $O$ , requires

$$\dot{\mathbf{H}}_O = \mathbf{M} \quad (9)$$

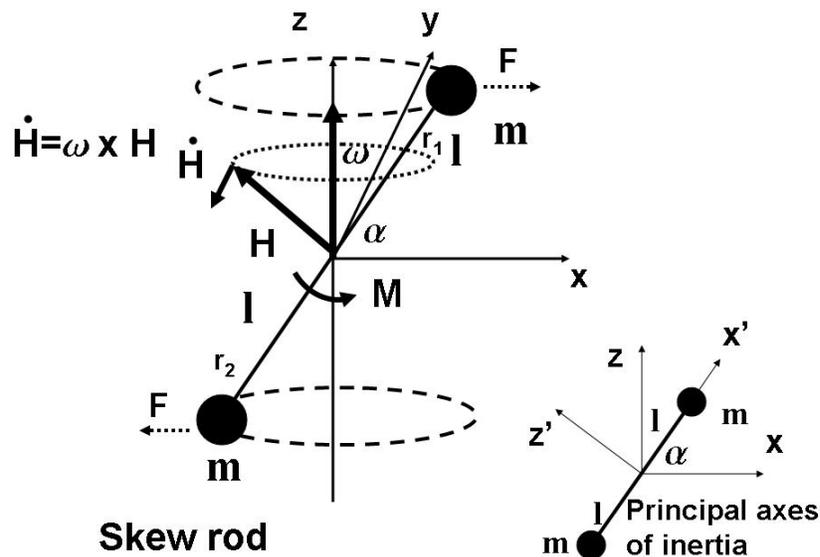
or about the center of mass  $G$

$$\dot{\mathbf{H}}_G = \mathbf{M}_G \quad (10)$$

We first examine cases in which the motion is specified and our task is to determine the forces and moments.

## Example: Rotating Skew Rod

We consider a simple example which illustrate the effect of rotation about a non-principal axis. In this case, we consider two masses equal  $m_1$  and  $m_2$  attached to a massless rigid rod of length  $2l$  aligned from the horizontal by an angle  $\alpha$  and rotating about the  $z$  axis with *constant* angular velocity  $\boldsymbol{\omega} = \omega\mathbf{k}$ . We examine this problem at the instant when the rod is aligned with the  $x$  and  $z$  axis. Because of symmetry, the center of mass is at the origin. Therefore, we consider only the change in angular momentum. Because the magnitude of the angular velocity  $\omega$  is constant, we are concerned only with its change in direction.



The angular momentum for this system is  $\mathbf{H} = \sum \mathbf{r} \times m_i \mathbf{v}_i$ . For this case, the velocity for mass  $m_1$  is  $\mathbf{v}_1 = \omega l \cos \alpha \mathbf{j}$ , while for mass  $m_2$  the velocity is  $\mathbf{v}_2 = -\omega l \cos \alpha \mathbf{j}$  while  $\mathbf{r}_1 = l \cos \alpha \mathbf{i} + l \sin \alpha \mathbf{k}$  and  $\mathbf{r}_2 = -l \cos \alpha \mathbf{i} - l \sin \alpha \mathbf{k}$ . At this instant of time, the angular momentum is in the  $x, z$  plane and of magnitude

$$\mathbf{H} = -2ml^2 \omega \cos \alpha \sin \alpha \mathbf{i} + 2ml^2 \omega \cos^2 \alpha \mathbf{k} \quad (11)$$

Since the axis about which the rotation takes place is not a principal axis, it is no surprise that the angular momentum vector is not aligned with the angular velocity vector. In this simple case, since we are dealing with two mass points, the rod has zero moment of inertia when rotate about its axis. Therefore the angular momentum vector is perpendicular to the rod. As the skew rod rotates about the  $z$  axis, the angular momentum changes as

$$\dot{\mathbf{H}} = \boldsymbol{\omega} \times \mathbf{H} = -2m\omega^2 l^2 \sin \alpha \cos \alpha \mathbf{j} \quad (12)$$

resulting in a "required" moment about the  $y$  axis to sustain the motion of

$$\mathbf{M} = -2m\omega^2 l^2 \sin \alpha \cos \alpha \mathbf{j} \quad (13)$$

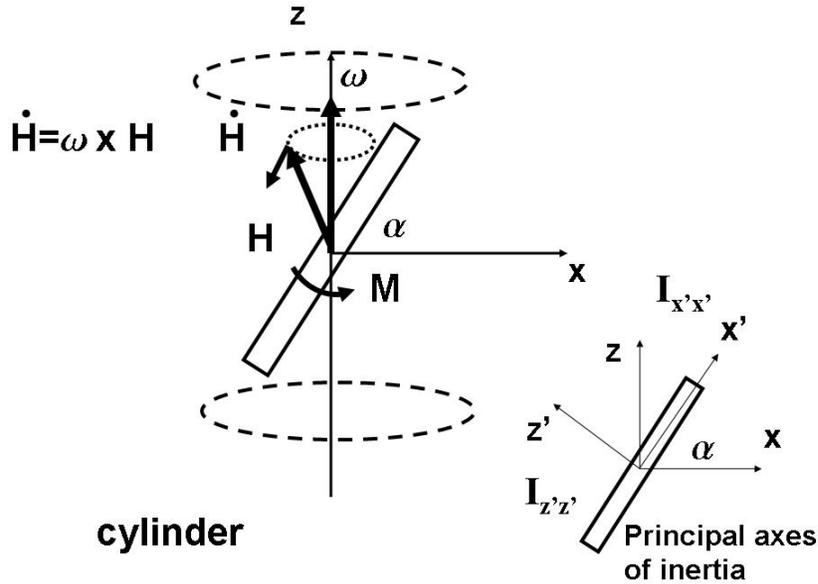
This moment has a physical interpretation in term of centripetal acceleration. As the rod rotates with angular velocity  $\boldsymbol{\omega}$ , the individual mass points experience a centripetal acceleration directed towards the  $z$  axis of  $\mathbf{a} = -\omega^2 l \cos \alpha \mathbf{i}$ . This requires an external force directed along the  $\mathbf{i}$  direction: on  $m_1$ , a force of  $\mathbf{F}_1 = -\omega^2 m l \cos \alpha \mathbf{i}$ ; on  $m_2$ , a force of  $\mathbf{F}_2 = \omega^2 m l \cos \alpha \mathbf{i}$ ; no net force is required because the body is supported at its center of mass. These forces can only be supplied/resisted by an external moment applied at the origin, where the rod is supported, a moment about the origin of  $\mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 = -2m\omega^2 l^2 \sin \alpha \cos \alpha \mathbf{j}$ , which is the moment required to sustain the motion.

As the rod rotates about the  $z$  axis, the moment or torque required to sustain the motion also rotates. Thus any fixture that is used to attach the rotating rod to the axle must sustain this torque.

Observe that this force goes to zero for  $\alpha = 0$  and  $\alpha = \pi/2$ . For these  $\alpha$ 's, the rod would be rotating about a principal axis.

### Example: Rotating Cylinder

The previous discussion can easily be extended, replacing the rod by a cylinder. In this case the moment of inertia  $I_{x'x'}$  is not equal to 0. Therefore, the angular momentum vector is not perpendicular to the cylinder but is at an angle determined by the relative values of  $I_{z'z'}$  and  $I_{x'x'}$ . If the body is not a "cylinder", i.e.  $I_{x'x'} \neq I_{y'y'} \neq I_{z'z'}$ , then the result is qualitatively similar but the angular momentum vector  $\mathbf{H}$  is not in the plane formed by the cylinder axis and  $\boldsymbol{\omega}$ .



### Example: Use of the inertia tensor

These examples may be treated using the more formal description in term of the inertia tensor. The principal axis of this system is clearly along the rod joining the two masses, as shown in the figure. If we treat the masses as mass points, there is no moment of inertia about the  $x'$  axis, while the moments of inertia about the  $y'$  and  $z'$  axis are equal and equal to  $I_{z'z'} = I_{y'y'} = 2ml^2$ . The coordinate transformation from the principal axes  $x', y'$  to the  $x, y$  axis is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \quad (14)$$

The inertia tensor in the  $x, y, z$  system is obtained by the transformation introduced in Lecture 26:  $[I] = [T][I'][T]^T$ .

$$\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2ml^2 & 0 \\ 0 & 0 & 2ml^2 \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}. \quad (15)$$

resulting in  $[I]$  for the inertia matrix in the  $x, y, z$  system as

$$[I] = 2ml^2 \begin{pmatrix} \sin^2\alpha & 0 & -\cos\alpha\sin\alpha \\ 0 & 1 & 0 \\ -\cos\alpha\sin\alpha & 0 & \cos^2\alpha \end{pmatrix}. \quad (16)$$

This confirms that the  $x$  and  $z$  axis are not principal axes. The angular momentum vector is then

$$\begin{pmatrix} H_{Gx} \\ H_{Gy} \\ H_{Gz} \end{pmatrix} = 2ml^2 \begin{pmatrix} \sin^2\alpha & 0 & -\cos\alpha\sin\alpha \\ 0 & 1 & 0 \\ -\cos\alpha\sin\alpha & 0 & \cos^2\alpha \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = 2ml^2\omega \begin{pmatrix} -\cos\alpha\sin\alpha \\ 0 \\ \cos^2\alpha \end{pmatrix} \quad (17)$$

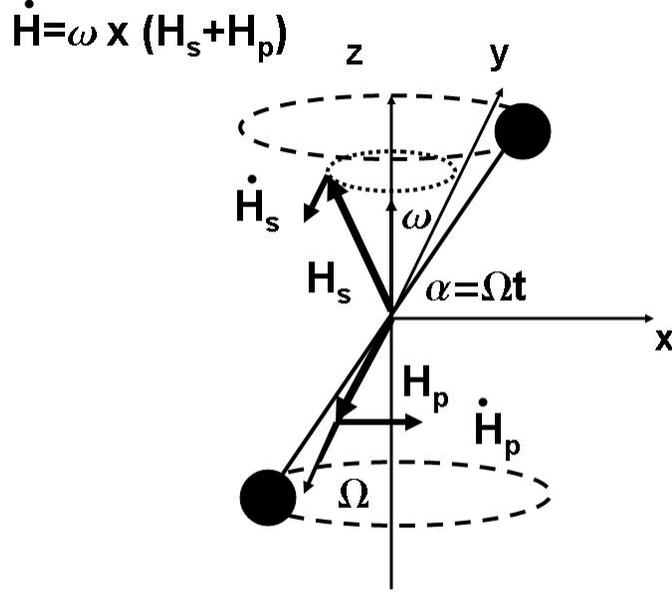
for the angular momentum. Leading to the required moment to sustain the motion as

$$\mathbf{M} = \dot{\mathbf{H}} = \boldsymbol{\omega} \times \mathbf{H} = -2l^2m\omega^2\cos\alpha\sin\alpha\mathbf{j} \quad (18)$$

in agreement with equation (13).

### Example: Two-Bladed Wind Turbines

An exciting application of the analysis of the rotating skew rod is to the operation of a two-bladed wind turbine. Recall that the analysis predicted that a constant moment about a horizontal axis is required from the support structure if the skew rod is to rotate about the vertical axis with angular velocity  $\omega$ . This analysis is also valid if  $\alpha$  is a function of time if we add the angular momentum about the  $y$  axis as a result of the "propeller" rotating about its own axis. Let  $\alpha = \Omega t$ , that is the rod rotates about the horizontal  $y$  axis through its center of mass and perpendicular to the plane containing the rod and masses, like a propeller. In addition, it rotates about the  $z$  axis following the previous analysis of the rotating skew rod. The total angular momentum is then the result from the previous analysis with  $\alpha = \Omega t$ ; the angular momentum of the skew rod becomes  $\mathbf{H}_s = -2ml^2\omega\cos(\Omega t)\sin(\Omega t)\mathbf{i} + 2ml^2\omega\cos^2(\Omega t)\mathbf{k}$  plus angular momentum due to the rotation of the propeller of magnitude  $\mathbf{H}_p = -2\Omega ml^2\mathbf{j}$ . The additional change in angular momentum due to the "propeller" rotation is simply  $\dot{\mathbf{H}}_p = \boldsymbol{\omega} \times \mathbf{H}_p$ . This is constant in time in a coordinate system rotating with the plane of the propeller. The motion requires an applied steady moment.



The total angular momentum of this configuration is then the sum of that for the skew rod plus that for the rotation of the propeller about the  $y$  axis. However, the additional change of angular momentum occurs only through the action of  $\omega \mathbf{k}$  in rotating the angular momentum of the propeller. The angular velocity of the propeller,  $\Omega$  does not rotate the angular momentum vector of the skew rod.

$$\mathbf{H} = \mathbf{H}_s + \mathbf{H}_p = -2l^2 m \omega \cos(\Omega t) \sin(\Omega t) \mathbf{i} - 2ml^2 \Omega \mathbf{j} + 2ml^2 \omega \cos^2(\Omega t) \mathbf{k} \quad (19)$$

Since only  $\omega \mathbf{k}$  rotates the angular momentum vector, we have from Coriolis theorem that the rate of change of angular momentum is

$$\dot{\mathbf{H}} = (\omega \mathbf{k}) \times (\mathbf{H}_s + \mathbf{H}_p) \quad (20)$$

Then the moment required is

$$\mathbf{M} = 2l^2 m (\omega \Omega \mathbf{i} - \omega^2 \cos(\Omega t) \sin(\Omega t) \mathbf{j} + 0 \mathbf{k}). \quad (21)$$

Note the unsteady behavior of the required moment with time. The moment required to produce the propeller rotation about  $\mathbf{j}$ , and to move the direction of this rotation about the  $z$  axis with angular velocity  $\omega$  is steady; the moment required to produce the rotation of the skew rod about  $\mathbf{k}$  is unsteady. This is due to the imbalance in inertia. At various times in its motion, the device is rotating about the  $z$  axis ( $\mathbf{k}$ ) which is not a principal axis for the configuration.

Consider this problem as a model for a two-bladed wind turbine. The wind turbine will happily rotate about its  $y$  axis with angular velocity  $\Omega$ . But wind turbines must yaw to face into a new wind direction. This requires for a short time an angular velocity  $\omega$  about the  $z$  axis. From our analysis, as the wind turbine

rotates in yaw to face the new wind direction, an oscillatory moment of  $\mathbf{M} = -2l^2m\omega^2\cos(\Omega t)\sin(\Omega t)\mathbf{j}$  will be exerted on the support. This undesirable oscillatory load will be eliminated if the wind turbine has three blades so that all directions in the  $x, z$  plane are principal axis. The figure shows how this idea has caught on.



Image by brentdanley on Flickr.

#### **ADDITIONAL READING**

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition  
5/1, 5/2, 5/3, 5/4 (review) , 5/5, 5/6 (review)

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16.07 Dynamics  
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