

Lecture L26 - 3D Rigid Body Dynamics: The Inertia Tensor

In this lecture, we will derive an expression for the angular momentum of a 3D rigid body. We shall see that this introduces the concept of the Inertia Tensor.

Angular Momentum

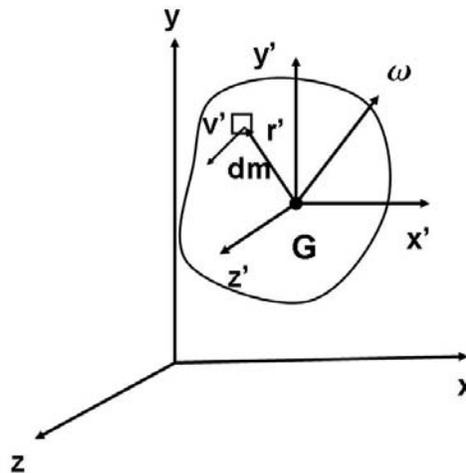
We start from the expression of the angular momentum of a system of particles about the center of mass, \mathbf{H}_G , derived in lecture L11,

$$\mathbf{H}_G = \sum_{i=1}^n (\mathbf{r}'_i \times m_i (\boldsymbol{\omega} \times \mathbf{r}'_i)) = \sum_{i=1}^n m_i r_i'^2 \boldsymbol{\omega} \quad (1)$$

$$\mathbf{H}_G = \sum_{i=1}^n (\mathbf{r}'_i \times m_i (\boldsymbol{\omega} \times \mathbf{r}'_i)) = \sum_{i=1}^n m_i r_i'^2 \boldsymbol{\omega} = \int_m \mathbf{r}' \times \mathbf{v}' dm \quad (2)$$

$$\mathbf{H}_G = \int_m \mathbf{r}' \times \mathbf{v}' dm .$$

Here, \mathbf{r}' is the position vector relative to the center of mass, \mathbf{v}' is the velocity relative to the center of mass. We note that, in the above expression, an integral is used instead of a summation, since we are now dealing with a continuum distribution of mass.



For a 3D rigid body, the distance between any particle and the center of mass will remain constant, and the particle velocity, relative to the center of mass, will be given by

$$\mathbf{v}' = \boldsymbol{\omega} \times \mathbf{r}' .$$

Thus, we have,

$$\mathbf{H}_G = \int_m \mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{r}') dm = \int_m [(\mathbf{r}' \cdot \mathbf{r}')\boldsymbol{\omega} - (\mathbf{r}' \cdot \boldsymbol{\omega})\mathbf{r}'] dm .$$

Here, we have used the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$. We note that, for planar bodies undergoing a 2D motion in its own plane, \mathbf{r}' is perpendicular to $\boldsymbol{\omega}$, and the term $(\mathbf{r}' \cdot \boldsymbol{\omega})$ is zero. In this case, the vectors $\boldsymbol{\omega}$ and \mathbf{H}_G are always parallel. In the three-dimensional case however, this simplification does not occur, and as a consequence, the angular velocity vector, $\boldsymbol{\omega}$, and the angular momentum vector, \mathbf{H}_G , are in general, *not parallel*.

In cartesian coordinates, we have, $\mathbf{r}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$ and $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$, and the above expression can be expanded to yield,

$$\begin{aligned} \mathbf{H}_G &= \left(\omega_x \int_m (x'^2 + y'^2 + z'^2) dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z') x' dm \right) \mathbf{i} \\ &+ \left(\omega_y \int_m (x'^2 + y'^2 + z'^2) dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z') y' dm \right) \mathbf{j} \\ &+ \left(\omega_z \int_m (x'^2 + y'^2 + z'^2) dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z') z' dm \right) \mathbf{k} \\ &= (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z) \mathbf{i} \\ &+ (-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z) \mathbf{j} \\ &+ (-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z) \mathbf{k} . \end{aligned} \quad (3)$$

The quantities I_{xx} , I_{yy} , and I_{zz} are called *moments of inertia* with respect to the x , y and z axis, respectively, and are given by

$$I_{xx} = \int_m (y'^2 + z'^2) dm , \quad I_{yy} = \int_m (x'^2 + z'^2) dm , \quad I_{zz} = \int_m (x'^2 + y'^2) dm .$$

We observe that the quantity in the integrand is precisely the square of the distance to the x , y and z axis, respectively. They are analogous to the moment of inertia used in the two dimensional case. It is also clear, from their expressions, that the moments of inertia are always positive. The quantities I_{xy} , I_{xz} , I_{yx} , I_{yz} , I_{zx} and I_{zy} are called *products of inertia*. They can be positive, negative, or zero, and are given by,

$$I_{xy} = I_{yx} = \int_m x' y' dm , \quad I_{xz} = I_{zx} = \int_m x' z' dm , \quad I_{yz} = I_{zy} = \int_m y' z' dm .$$

They are a measure of the imbalance in the mass distribution. If we are interested in calculating the angular momentum with respect to a fixed point O then, the resulting expression would be,

$$\begin{aligned} \mathbf{H}_O &= ((I_{xx})_O \omega_x - (I_{xy})_O \omega_y - (I_{xz})_O \omega_z) \mathbf{i} \\ &+ (-(I_{yx})_O \omega_x + (I_{yy})_O \omega_y - (I_{yz})_O \omega_z) \mathbf{j} \\ &+ (-(I_{zx})_O \omega_x - (I_{zy})_O \omega_y + (I_{zz})_O \omega_z) \mathbf{k} . \end{aligned} \quad (4)$$

Here, the moments of products of inertia have expressions which are analogous to those given above but with x' , y' and z' replaced by x , y and z . Thus, we have that

$$(I_{xx})_O = \int_m (y^2 + z^2) dm, \quad (I_{yy})_O = \int_m (x^2 + z^2) dm, \quad (I_{zz})_O = \int_m (x^2 + y^2) dm,$$

and,

$$(I_{xy})_O = (I_{yx})_O = \int_m xy dm, \quad (I_{xz})_O = (I_{zx})_O = \int_m xz dm, \quad (I_{yz})_O = (I_{zy})_O = \int_m yz dm.$$

The Tensor of Inertia

The expression for angular momentum given by equation (3), can be written in matrix form as,

$$\begin{pmatrix} H_{Gx} \\ H_{Gy} \\ H_{Gz} \end{pmatrix} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}. \quad (5)$$

or,

$$\mathbf{H}_G = [I_G] \boldsymbol{\omega}, \quad (6)$$

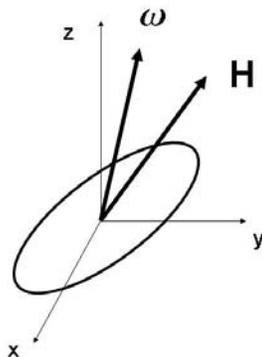
where $[I_G]$ is the tensor of inertia (written in matrix form) about the center of mass G and with respect to the xyz axes. The tensor of inertia gives us an idea about how the mass is distributed in a rigid body.

Analogously, we can define the tensor of inertia about point O , by writing equation(4) in matrix form. Thus, we have

$$\mathbf{H}_O = [I_O] \boldsymbol{\omega},$$

where the components of $[I_O]$ are the moments and products of inertia about point O given above.

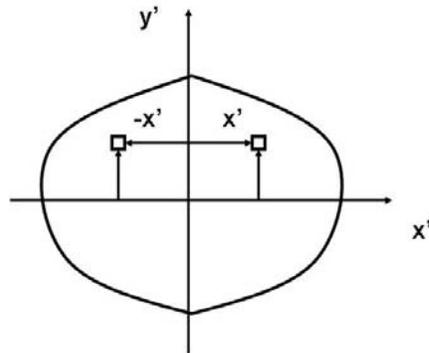
It follows from the definition of the products of inertia, that the tensors of inertia are always *symmetric*. The implications of equation (5) are that in many situations of importance, even for bodies of some symmetry, the angular momentum vector \vec{H} and the angular velocity vector $\vec{\omega}$ are not parallel. This introduces considerable complexity into the analysis of the dynamics of rotating bodies in three dimensions.



Principal Axes of Inertia

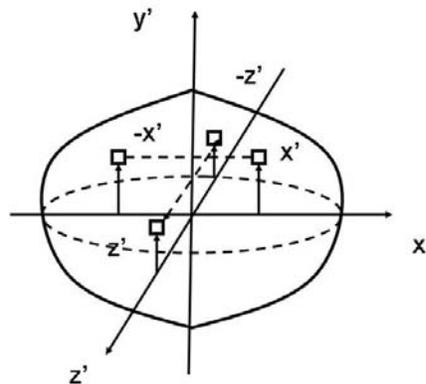
For a general three-dimensional body, it is *always* possible to find 3 mutually orthogonal axis (an x, y, z coordinate system) for which the products of inertia are zero, and the inertia matrix takes a diagonal form. In most problems, this would be the preferred system in which to formulate a problem. For a rotation about only one of these axis, the angular momentum vector is parallel to the angular velocity vector. For symmetric bodies, it may be obvious which axis are principle axis. However, for an irregular-shaped body this coordinate system may be difficult to determine by inspection; we will present a general method to determine these axes in the next section.

But, if the body has symmetries with respect to some of the axis, then some of the products of inertia become zero and we can identify the principal axes. For instance, if the body is symmetric with respect to the plane $x' = 0$ then, we will have $I_{x'y'} = I_{y'x'} = I_{x'z'} = I_{z'x'} = 0$ and x' will be a principal axis. This can be shown by looking at the definition of the products of inertia.

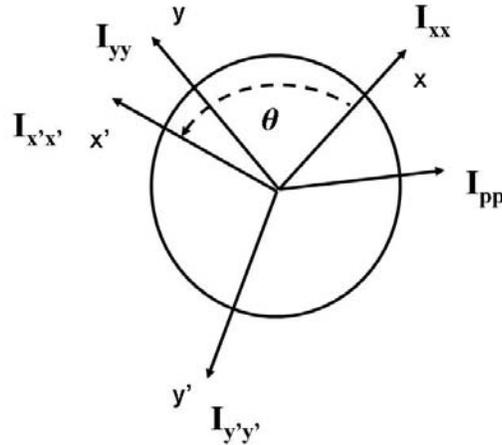


The integral for, say, $I_{x'y'}$ can be decomposed into two integrals for the two halves of the body at either side of the plane $x' = 0$. The integrand on one half, $x'y'$, will be equal in magnitude and opposite in sign to the integrand on the other half (because x' will change sign). Therefore, the integrals over the two halves will cancel each other and the product of inertia $I_{x'y'}$ will be zero. (As will the product of inertia $I_{x'z'}$)

Also, if the body is symmetric with respect to two planes passing through the center of mass which are orthogonal to the coordinate axis, then the tensor of inertia is diagonal, with $I_{x'y'} = I_{x'z'} = I_{y'z'} = 0$.



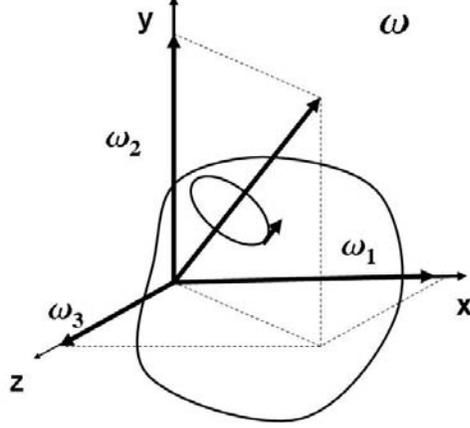
Another case of practical importance is when we consider axisymmetric bodies of revolution. In this case, if one of the axis coincides with the axis of symmetry, the tensor of inertia has a simple diagonal form. For an axisymmetric body, the moments of inertia about the two axis in the plane will be equal. Therefore, the moment about any axis in this plane is equal to one of these. And therefore, any axis in the plane is a principal axis. One can extend this to show that if the moment of inertia is equal about two axis in the plane ($I_{PP} = I_{xx}$), whether or not they are orthogonal, then all axes in the plane are principal axes and the moment of inertia is the same about all of them. In its inertial properties, the body behaves like a circular cylinder.



The tensor of inertia will take different forms when expressed in different axes. When the axes are such that the tensor of inertia is diagonal, then these axes are called the *principal axes of inertia*.

The Search for Principal Axes and Moments of Inertia as an Eigenvalue Problem

Three orthogonal principal axes of inertia always exist even though in bodies without symmetries their directions may not be obvious. To find the principle axis of a general body consider the body shown in the figure that rotates about an unknown principal axis. The the total angular momentum vector is $I\vec{\omega}$ in the direction of the principle axis. For rotation about the principal axis, the angular momentum and the angular velocity are in the same direction.



We seek a coordinate axes x , y and z , about which a rotation ω_x , ω_y and ω_z , which is aligned with this coordinate direction, will be parallel to the angular momentum vector and related by the equation

$$\begin{pmatrix} H_{Gx} \\ H_{Gy} \\ H_{Gz} \end{pmatrix} = \begin{pmatrix} I\omega_x \\ I\omega_y \\ I\omega_z \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}. \quad (7)$$

We then express the **general** form for angular momentum vector in components along the x , y and z axis in term of the components of $\vec{\omega}$ along these axes using the **general** form of the inertia tensor in the x, y, z system, we have

$$\begin{pmatrix} H_{Gx} \\ H_{Gy} \\ H_{Gz} \end{pmatrix} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}. \quad (8)$$

To obtain the special directions of ω that is aligned with a principal axis, we equate these two expressions.

$$\begin{pmatrix} H_{Gx} \\ H_{Gy} \\ H_{Gz} \end{pmatrix} = \begin{pmatrix} I\omega_x \\ I\omega_y \\ I\omega_z \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}. \quad (9)$$

At this point in the process we know the inertia tensor in an arbitrary x , y , and z system and are seeking the special orientation of ω which will align the angular momentum H_G with the angular velocity ω . Collecting terms from equation(11) on the left-hand side, we obtain

$$\begin{pmatrix} (I_{xx} - I) & -I_{xy} & -I_{xz} \\ -I_{yx} & (I_{yy} - I) & -I_{yz} \\ -I_{zx} & -I_{zy} & (I_{zz} - I) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (10)$$

resulting in the requirement that

$$\left(\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} - I \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (11)$$

The structure of the solution for finding the principal axes of inertia and their magnitudes is a **characteristic-value problem**. The three eigenvalues give the directions of the three principal axis, and the three eigenvectors give the moments of inertia with respect to each of these axis.

In principal directions, the inertia tensor has the form

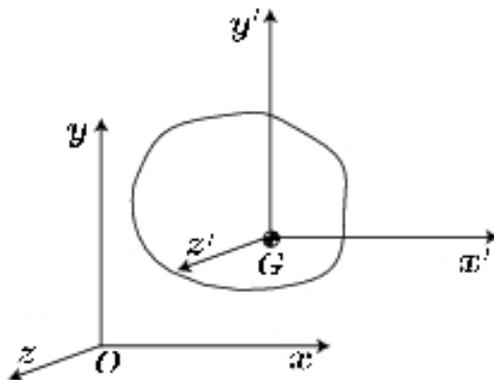
$$[I_G] = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

where we will write $I_x = I_{xx}$, $I_y = I_{yy}$ and $I_z = I_{zz}$. Also, in principal axes we will then have

$$\mathbf{H}_G = I_x \omega_x \mathbf{i} + I_y \omega_y \mathbf{j} + I_z \omega_z \mathbf{k} .$$

Parallel Axis Theorem

It will often be easier to obtain the tensor of inertia with respect to axis passing through the center of mass. In some problems however, we will need to calculate the tensor of inertia about different axes. The parallel axis theorem introduced in lecture L22 for the two dimensional moments of inertia can be extended and applied to each of the components of the tensor of inertia.



In particular we can write,

$$\begin{aligned} (I_{xx})_O &= \int_m (y^2 + z^2) dm = \int_m ((y_G + y')^2 + (z_G + z')^2) dm \\ &= \int_m (y'^2 + z'^2) + 2y_G \int_m y' dm + 2z_G \int_m z' dm + (y_G^2 + z_G^2) \int_m dm \\ &= I_{xx} + m(y_G^2 + z_G^2) . \end{aligned}$$

Here, we have use the fact that y' and z' are the coordinates relative to the center of mass and therefore their integrals over the body are equal to zero. Similarly, we can write,

$$(I_{yy})_O = I_{yy} + m(x_G^2 + z_G^2), \quad (I_{zz})_O = I_{zz} + m(x_G^2 + y_G^2),$$

and,

$$(I_{xy})_O = (I_{yx})_O = I_{xy} + mx_G y_G, \quad (I_{xz})_O = (I_{zx})_O = I_{xz} + mx_G z_G, \quad (I_{yz})_O = (I_{zy})_O = I_{yz} + my_G z_G .$$

Rotation of Axes

In some situations, we will know the tensor of inertia with respect to some axes xyz and, we will be interested in calculating the tensor of inertia with respect to another set of axis $x'y'z'$. We denote by \mathbf{i} , \mathbf{j} and \mathbf{k} the unit vectors along the direction of xyz axes, and by \mathbf{i}' , \mathbf{j}' and \mathbf{k}' the unit vectors along the direction of $x'y'z'$ axes. The transformation of the inertia tensor can be accomplished by considering the transformation of the angular momentum vector \vec{H} and the angular velocity vector $\vec{\omega}$. We begin with the expression of \vec{H} and $\vec{\omega}$ in the x_1, x_2, x_3 system.

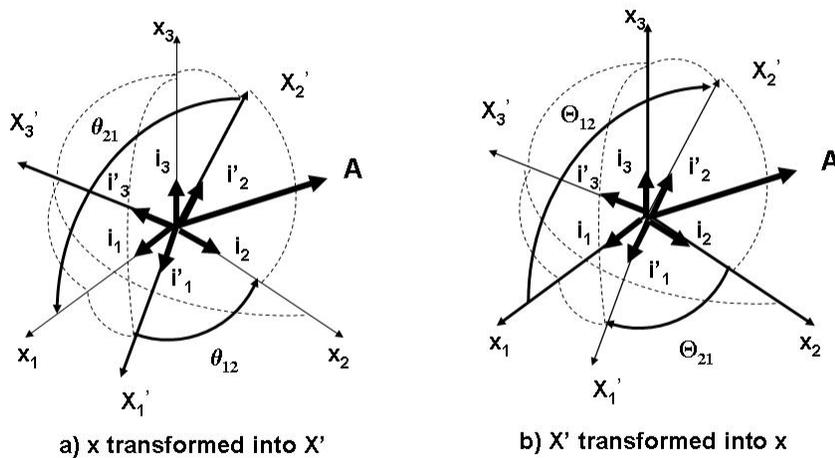
$$\vec{H} = [I]\vec{\omega} \quad (12)$$

We have not indicated by a subscript where the origin of our coordinates are, i.e. the center of mass G , a fixed point O , or any other point, because as long as we are simply doing a rotational transformation of coordinates about this point, it does not matter,

From Lecture 3, we have that the transformation of a vector from a coordinate system x_1, x_2, x_3 into a coordinate system x'_1, x'_2, x'_3 is given by

$$\begin{pmatrix} H'_1 \\ H'_2 \\ H'_3 \end{pmatrix} = \begin{pmatrix} \mathbf{i}'_1 \cdot \mathbf{i}_1 & \mathbf{i}'_1 \cdot \mathbf{i}_2 & \mathbf{i}'_1 \cdot \mathbf{i}_3 \\ \mathbf{i}'_2 \cdot \mathbf{i}_1 & \mathbf{i}'_2 \cdot \mathbf{i}_2 & \mathbf{i}'_2 \cdot \mathbf{i}_3 \\ \mathbf{i}'_3 \cdot \mathbf{i}_1 & \mathbf{i}'_3 \cdot \mathbf{i}_2 & \mathbf{i}'_3 \cdot \mathbf{i}_3 \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = [T] \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}.$$

where we have introduced the symbol $[T]$ for the transformation matrix.



Likewise, the transformation of $\vec{\omega}$ into the x'_1, x'_2, x'_3 system is given by

$$\begin{pmatrix} \omega'_1 \\ \omega'_2 \\ \omega'_3 \end{pmatrix} = \begin{pmatrix} \mathbf{i}'_1 \cdot \mathbf{i}_1 & \mathbf{i}'_1 \cdot \mathbf{i}_2 & \mathbf{i}'_1 \cdot \mathbf{i}_3 \\ \mathbf{i}'_2 \cdot \mathbf{i}_1 & \mathbf{i}'_2 \cdot \mathbf{i}_2 & \mathbf{i}'_2 \cdot \mathbf{i}_3 \\ \mathbf{i}'_3 \cdot \mathbf{i}_1 & \mathbf{i}'_3 \cdot \mathbf{i}_2 & \mathbf{i}'_3 \cdot \mathbf{i}_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = [T] \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.$$

To determine \vec{H}' (in the transformed system) we multiply both sides of equation (10) by $[T]$, and obtain

$$\vec{H}' = [T]\vec{H} = [T][I]\vec{\omega} \quad (13)$$

where $[I]$ is the inertia matrix in the original coordinate system. But this equation is not quite in the proper form; we need to have the form $\vec{H}' = [I']\vec{\omega}'$ to identify $[I']$, the inertia matrix in the new coordinate system. We take advantage of the fact that the matrix $[T]^T$ is also the inverse of $[T]$, $[T]^{-1}$, so that $[T]^T [T] = [Iden]$, where $[Iden]$ is the identity matrix. Now, we can always stick an identity matrix in a matrix equation—like multiplying by 1—so that we may write

$$(\vec{H}') = [T]\vec{H} = [T][I][T]^T [T]\vec{\omega} = ([T][I][T]^T)([T]\vec{\omega}) = [I']\omega' \quad (14)$$

were we have grouped the terms so that it is obvious that the inertia tensor in the transformed coordinate system is

$$[I'] = [T][I][T]^T \quad (15)$$

Therefore, if we want to calculate the tensor of inertia with respect to axis $x'y'z'$, we can write in matrix form

$$\begin{pmatrix} I_{x'x'} & -I_{x'y'} & -I_{x'z'} \\ -I_{y'x'} & I_{y'y'} & -I_{y'z'} \\ -I_{z'x'} & -I_{z'y'} & I_{z'z'} \end{pmatrix} = \begin{pmatrix} \mathbf{i}' \cdot \mathbf{i} & \mathbf{i}' \cdot \mathbf{j} & \mathbf{i}' \cdot \mathbf{k} \\ \mathbf{j}' \cdot \mathbf{i} & \mathbf{j}' \cdot \mathbf{j} & \mathbf{j}' \cdot \mathbf{k} \\ \mathbf{k}' \cdot \mathbf{i} & \mathbf{k}' \cdot \mathbf{j} & \mathbf{k}' \cdot \mathbf{k} \end{pmatrix} \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{i} \cdot \mathbf{i}' & \mathbf{j} \cdot \mathbf{i}' & \mathbf{k} \cdot \mathbf{i}' \\ \mathbf{i} \cdot \mathbf{j}' & \mathbf{j} \cdot \mathbf{j}' & \mathbf{k} \cdot \mathbf{j}' \\ \mathbf{i} \cdot \mathbf{k}' & \mathbf{j} \cdot \mathbf{k}' & \mathbf{k} \cdot \mathbf{k}' \end{pmatrix}.$$

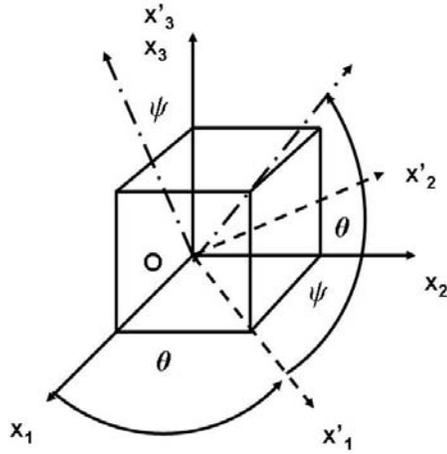
where we have returned to the x, y, z notation.

Likewise, expressed in the direction cosines of the figure, the transformation is

$$[I'] = \begin{pmatrix} \cos(\theta_{11}) & \cos(\theta_{12}) & \cos(\theta_{13}) \\ \cos(\theta_{21}) & \cos(\theta_{22}) & \cos(\theta_{23}) \\ \cos(\theta_{31}) & \cos(\theta_{32}) & \cos(\theta_{33}) \end{pmatrix} \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \cos(\Theta_{11}) & \cos(\Theta_{12}) & \cos(\Theta_{13}) \\ \cos(\Theta_{21}) & \cos(\Theta_{22}) & \cos(\Theta_{23}) \\ \cos(\Theta_{31}) & \cos(\Theta_{32}) & \cos(\Theta_{33}) \end{pmatrix} \dots \quad (16)$$

Inertia Tensor of a Cube

We are going to consider and calculate the inertia tensor of a cube of equal length sides b . (This example is taken from Marion and Thorton.) For this choice of coordinates, it is obvious that the center of mass does not lie at the origin. It is also obvious that there is considerable symmetry in the geometry.



Evaluating the various integrals for the components of the inertia tensor, we have

$$[I] = \begin{pmatrix} \frac{2}{3}Mb^2 & -\frac{1}{4}Mb^2 & -\frac{1}{4}Mb^2 \\ -\frac{1}{4}Mb^2 & \frac{2}{3}Mb^2 & -\frac{1}{4}Mb^2 \\ -\frac{1}{4}Mb^2 & -\frac{1}{4}Mb^2 & \frac{2}{3}Mb^2 \end{pmatrix} \quad (17)$$

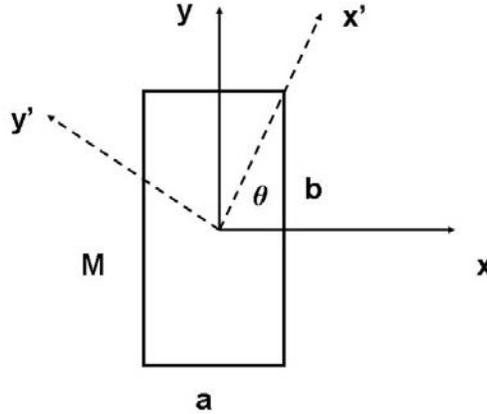
This result is perhaps a bit surprising; because of the strong symmetry of the cube we might have expected these coordinates to be principal axes. However, our origin of coordinates is not at the center of mass, and by the parallel axis theorem, we incur products of inertia equal to total mass times the various distances of center of mass from the origin of coordinates.

There are several options to find a set of principal axes for this problem. One is to use the symmetry to identify a coordinate system in which the products of inertia vanish. The transformation sketched in the figure is a good candidate to obtain the principal axes; that is a rotation of $\theta = \pi/4$ about the x_3 axis, followed by a rotation of $\psi = \sin^{-1} \frac{1}{\sqrt{3}}$ about the x'_1 axis. This would result in the x_1 axis going through the upper point of the cube and through the center of mass as well. If we were to do this, we would find that indeed this transformation results in coordinate directions which are principal axis directions, and moreover that the inertias are equal about any axis in the plane perpendicular to the diagonal of the cube, so that all these axes are principal with equal moments of inertia.

Another more general method is to express the search for principal axes as an eigenvalue problem, as previously outlined. In a problem without strong and obvious symmetry, identifying an appropriate transformation is often a matter of guess work; therefore the eigenvalue formulation is more useful.

Products of Inertia in a Plane

In many situations, we will know one principal axis from symmetry, and will be working with moments of inertia and inertia products in a plane. An example would be a flat plate of mass m , thickness t , width a and length b . It is clear from the sketch that the principal axes are the x, y coordinates. We may have reasons for wanting to express the moments of inertia and the products of inertia in the x', y' system (rotated from the x, y system by an angle θ) or to move back and forth between the x, y system, and the x', y' system.



The coordinate transformation that takes a vector in the x, y, z system into the x', y', z' system is

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (18)$$

or $(\vec{x}') = [T](\vec{x})$; while the coordinate transformation that takes a vector in the x', y', z' system into the x, y, z system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \quad (19)$$

or $(\vec{x}) = [T]^T(\vec{x}')$ where we have taken advantage of the fact that the matrix for the reverse transformation (\vec{x}' to \vec{x}) is the transpose of the matrix for the forward transformation (\vec{x} to \vec{x}'). The inertia matrix for a rectangular plate with its center of mass at the origin in the x, y, z system, for which the principal axes line up with the coordinate directions is

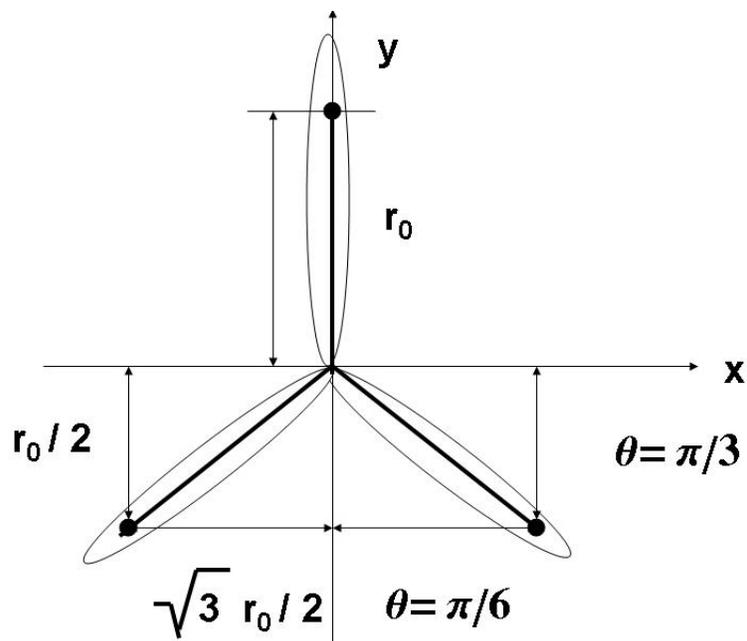
$$I_{x,y,z} = M \begin{pmatrix} \frac{b^2}{12} & 0 & 0 \\ 0 & \frac{a^2}{12} & 0 \\ 0 & 0 & \frac{a^2+b^2}{12} \end{pmatrix}. \quad (20)$$

If we wish to obtain the inertia tensor in the x', y', z' system, we apply the transformation

$$I_{x',y',z'} = [T][I_{x,y,z}][T]^T \quad (21)$$

The Symmetry of Three

An interesting and practical result for inertias in a plane comes from considering the moment of inertia of a body with tri-symmetry, such as a three-bladed propeller or a wind turbine. It is clear that the axis normal to the figure is a principal axis, with masses symmetrically balanced about the origin. It is also clear that both the x and y axis are principal axis, with inertia products going to zero by symmetry. What is perhaps surprising, is that I_{xx} the moment of inertia about the x axis is equal to I_{yy} the moment of inertia about the y axis.



Consider the x axis; for a point r_0 the contribution to the moment of inertia from all 3 mass points is $I_{xx} = m(r_0^2 + 2(r_0/2)^2) = m\frac{3r_0^2}{2}$. Consider the y axis; for a point r_0 the contribution to the moment of inertia from all 3 mass points is $I_{yy} = m(2(\sqrt{3}r_0/2)^2) = m\frac{3r_0^2}{2}$. Therefore, the total moment of inertia about both the x and y axis will be an integral of the mass distribution in dm times $\frac{3r_0^2}{2}$. Since the moments of inertia are equal about the 2 principal axis x and y, the moment of inertia about **any axis** in the x,y plane is also equal to I_{xx} . The three-bladed propeller behaves as a uniform disk. This has important implications for the vibrations of propellers of historically-interesting high performance propeller fighter aircraft and wind turbines in yawing motion.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

7/7, Appendix B

S. T. Thorton and J. B. Marion, *Classical Dynamics of Particles and Systems*, 4th Edition, Chapter 11

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16.07 Dynamics
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