J. Peraire, S. Widnall
16.07 Dynamics
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Lecture L23 - 2D Rigid Body Dynamics: Impulse and Momentum

In lecture L9, we saw the principle of impulse and momentum applied to particle motion. This principle was of particular importance when the applied forces were functions of time and when interactions between particles occurred over very short times, such as with impact forces. In this lecture, we extend these principles to two dimensional rigid body dynamics.

Impulse and Momentum Equations

Linear Momentum

In lecture L22, we introduced the equations of motion for a two dimensional rigid body. The linear momentum for a system of particles is defined as

$$\boldsymbol{L} = m\boldsymbol{v}_G \ ,$$

where m is the total mass of the system, and $\boldsymbol{v}_G$ is the velocity of the center of mass measured with respect to an inertial reference frame. Assuming the mass of the system to be constant, we have that the sum of the external applied forces to the system, $\boldsymbol{F}$, must equal the change in linear momentum, $\dot{\boldsymbol{L}} = \boldsymbol{F}$, or, integrating between times t_1 and t_2,

$$\boldsymbol{L}_2 - \boldsymbol{L}_1 = \int_{t_1}^{t_2} \boldsymbol{F} \, dt \qquad (1)$$

Note that this is a vector equation and therefore must be satisfied for each component separately. Expression 1 is particularly useful when the precise time variation of the applied forces is unknown, but their total impulse can be calculated. Of course, when the impulse of the applied forces is zero, the momentum is conserved and we have $(\boldsymbol{v}_G)_2 = (\boldsymbol{v}_G)_1$.

Angular Momentum

A similar expression to 1 can be derived for the angular momentum if we start from the principle of conservation of angular momentum, $\dot{\boldsymbol{H}}_G = \boldsymbol{M}_G$. Here, $\boldsymbol{H}_G = I_G\omega$ is the angular momentum about the center of mass, and $\boldsymbol{M}_G$ is the moment of all externally applied forces about the center of mass. Integrating between times t_1 and t_2, we have,

$$(\boldsymbol{H}_G)_2 - (\boldsymbol{H}_G)_1 = \int_{t_1}^{t_2} \boldsymbol{M}_G \, dt \ . \qquad (2)$$

In a similar manner, for rotation about a fixed point O, we can write,

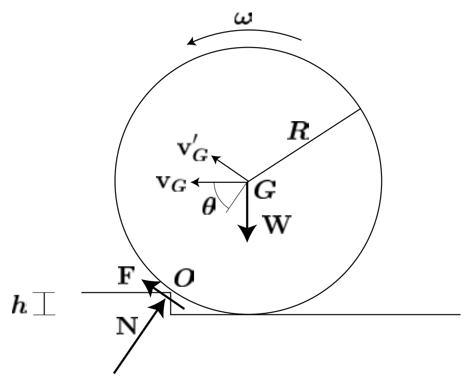
$$(\boldsymbol{H}_O)_2 - (\boldsymbol{H}_O)_1 = \int_{t_1}^{t_2} \boldsymbol{M}_O \, dt \quad , \tag{3}$$

where $\boldsymbol{H}_O = I_O \boldsymbol{\omega}$, the moment of inertia, I_O, refers to the fixed point O, and the external moments are with respect to point O.

Finally, if the external applied moment is zero, then we have conservation of angular momentum, which implies $\omega_2 = \omega_1$.

Example ***Cylinder rolling over a step***

We consider a disk of radius R rolling over a flat surface, and striking a rough step of height h.



Initial Impact

First, we consider the situation an instant before and an instant after the impact with the step. Before the impact, the velocity of the center of mass, $\boldsymbol{v}_G$, is in the horizontal direction and has a magnitude of $v_G = \omega R$. After the cylinder hits the step, it starts turning around point O, (the notation O will be used throughout this discussion for the instantaneous point of contact) and therefore the velocity of the center of mass, $\boldsymbol{v}_G$, must be perpendicular to the segment OG. Clearly, this change in velocity direction happens over a very short time interval and is caused by the impulsive forces generated during the impact. Since we are assuming a rough step, the point on the cylinder which is in contact with the step at O has zero velocity (no slipping) during this short interval. Therefore, we can use equation 3 about point O to characterize the change in $\boldsymbol{v}_G$. The angular momentum of the cylinder about O before impact is

$$H_O = I_G \omega + mR \sin\theta \, v_G = [\frac{I_G}{R} + m(R-h)] \, v_G \ .$$

The angular momentum just after impact is,

$$H'_O = I_O \omega' = I_O \frac{v'_G}{R} = [\frac{I_G}{R} + mR] \, v'_G \ .$$

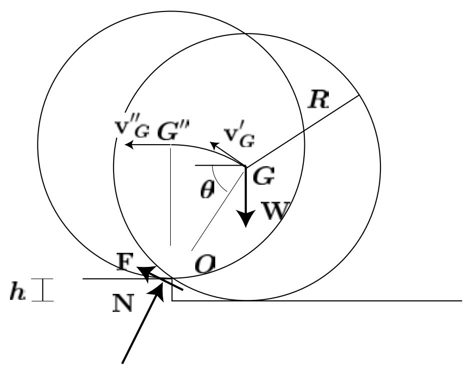
In these equations, ω and ω' are the angular velocities before and after the impact, respectively, and m is the mass of the cylinder. The change in angular momentum, according to equation 3, is equal to the angular impulse about O. Clearly, the normal and tangential forces, N and F, generate no moment about O. The only other force is the weight, $\boldsymbol{W}$, which is not an impulsive force. Hence, the impulse generated over a very short time interval, $t_2 \rightarrow t_1$, will be negligible. Thus, $H_O = H'_O$, which implies that,

$$v'_G = \frac{I_G + mR(R-h)}{I_G + mR^2} v_G = \frac{I_G + mR(R-h)}{I_O} v_G = (1 - \frac{Rh}{k_O^2})\, v_G \ , \tag{4}$$

where $k_O = \sqrt{I_O/m}$ is the radius of gyration of the cylinder about O.

Rolling about O

After the initial impact, the cylinder rolls about point O. During this process, since there are no dissipation mechanisms, energy will be conserved. Therefore, we can use the conservation of energy principle and require that the sum of the potential and kinetic energies remains constant.



The change in potential energy, mgh, is obtained by a decrease in the kinetic energy, thus

$$\frac{1}{2} I_O {\omega'}^2 = mgh + \frac{1}{2} I_O {\omega''}^2 \quad ,$$

or,

$${v''}_G^2 = {v'}_G^2 - 2gh \frac{R^2}{k_O^2} \quad .$$

The residual velocity after the cylinder has climbed the step, v''_G, can be expressed in terms of the initial velocity, v_G, using 4, as,

$${v''}_G^2 = (1 - \frac{Rh}{k_O^2})^2 v_G^2 - 2gh \frac{R^2}{k_O^2} \quad .$$

If we want to determine the minimum initial velocity, $(v_G)_{min}$, that would allow the cylinder to climb the step, we set the residual velocity, v''_G, to zero and obtain,

$$(v_G)^2_{min} = \frac{2gh \frac{R^2}{k_O^2}}{(1 - \frac{Rh}{k_O^2})^2} \quad .$$

Rebounding

So far, we have assumed that the cylinder pivots around point O without loosing contact with the step. It is clear that if the velocity v'_G is very large, then, as soon as the turning starts, a force will be required to produce the centripetal acceleration. For small velocities, the weight should be sufficient, but, for larger velocities, it is possible that the normal reaction force, N, will become zero, and the cylinder will loose contact with the step. The normal force, N, can be calculated from the normal momentum equation. That is, the sum of the forces in the direction of OG should equal the centripetal acceleration,

$$W \sin\theta - N = m\frac{{v'_G}^2}{R} \ .$$

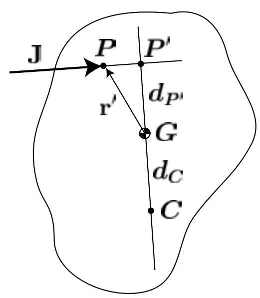
Setting $W = mg$, $\sin\theta = (R-h)/R$, and $N = 0$, we can determine the minimum velocity, $(v'_G)_{rebound}$, that will produce separation. The resulting expression can be combined with equation 4 to obtain an expression for the minimum initial velocity required for separation,

$$(v_G)^2_{rebound} = \frac{g(R-h)}{(1-\frac{Rh}{k_O^2})^2} \ .$$

We note that for $h/(R-h) < k_O^2/(2R^2)$, $(v_G)_{min} < (v_G)_{rebound}$, and, in this case, there is a range of velocities for which it is possible for the cylinder to climb the step without rebounding.

Center of Percussion Relative to an Instantaneous Center of Motion

In some situations, it is of interest to determine how a body should be impulsively set in motion such that a certain prescribed point–the point C–will be (at least momentarily) the instantaneous center of motion.



Consider a rigid body of mass m which is initially at rest. An impulse $\boldsymbol{J}$ is applied at $t = 0$ at point P in the body. The application of $\boldsymbol{J}$ to the body initiates both translational and rotational motion. Thus,

$$\begin{aligned} m\boldsymbol{v}_G &= \boldsymbol{J} \ , \\ I_G\boldsymbol{\omega} &= \boldsymbol{r}' \times \boldsymbol{J} \ . \end{aligned}$$

The modulus of the initial angular velocity is $\omega = d_{P'}J/I_G$, and the modulus of the velocity of the center of mass is $v_G = J/m$. We now wish to find out the point P such that a prescribed point C in the body is the instantaneous center of motion. If C is the center of motion, then $\boldsymbol{v}_G = \boldsymbol{\omega} \times \boldsymbol{r}_{CG}$, or, in magnitude $v_G = \omega d_C$. Therefore,

$$d_C = \frac{v_G}{\omega} = \frac{J/m}{d_{P'}J/I_G} = \frac{I_G}{md_{P'}} \ ,$$

or,

$$d'_P = \frac{I_G}{md_C} = \frac{k_G^2}{d_C} \ .$$

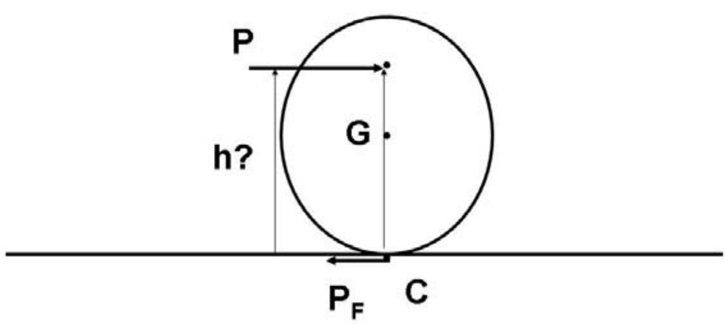
or

$$d_C d'_P = k_G^2$$

in other words, for any body, the product of d'_P the distance betweens the point of application of an impulse and d_C, the center of motion is a property of the body, the radius of gyration squared. Note that, since the impulsive moment is a vector, the same result would be obtained if the impulse $\boldsymbol{J}$ were applied anywhere along the line through P' in the direction of $\boldsymbol{J}$. The point P' is called the *center of percussion* associated with the center of motion C.

Example ***Striking a billiard ball***

Consider a billiard ball resting on a table. In the general case, if we apply an impulse at a point h above the table, there will be a reaction friction impulse with the table and the ball will roll. However, this process is unreliable, in that the coefficient of friction might not be large enough to produce a smooth motion. For improved accuracy we want to hit the ball at a height h such that ***no fiction force*** is required. Therefore we want to know at what height above the table we have to hit a billiard ball so that no friction impulse occurs and the ball rolls on the table without slipping.



The moment of inertia of a homogeneous sphere about its center of mass is $I_G = (2/5)mR^2$. We take moments about the instantaneous center of rotation, the contact point C between the ball and the table. By the parallel axis theorem, the moment of inertia about C is $I_C = 7/5MR_0^2$. The governing equations are: 1) relate linear impulse to the change in linear momentum for the center of mass, $MV_G = P - P_F$; 2) relate the moment of the impulse to the change in angular momentum taken about the point O, $Ph = I_O\omega$; 3) use the geometric relationship, $v_G = -R_0\omega$. We obtain the following result for the friction force required if the ball is to roll smoothly on the table.

$$P_F = P(1 - h\frac{5}{7R_0}) \tag{5}$$

therefore, if $h = \frac{7}{5}R_0$, no friction force will occur and the ball will more reliably roll smoothly. For our earlier discussion of center of percussion, we see that the point h is the center of percussion relative to the contact (instantaneous center of rotation) point O.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

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References

[1] M. Martinez-Sanchez, *Unified Engineering Notes*, Course 95-96.