

# Lecture L18 - Exploring the Neighborhood: the Restricted Three-Body Problem

## The Three-Body Problem

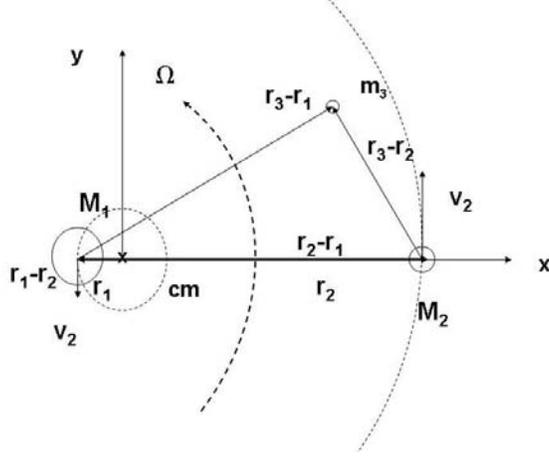
In Lecture 15-17, we presented the solution to the two-body problem for mutual gravitational attraction. We were able to obtain this solution in closed form. For bodies of comparable mass, the solution showed that the bodies orbited about their center of mass, with orbital periods that could be obtained analytically by transforming the two-body problem into a "Kepler" problem with modified gravitational forces. For a body of negligible mass, such as a satellite orbiting the earth, we were able to place the earth at rest at the center of our coordinates, and obtain a simple expression for the orbit of the satellite.

No such solution is available for the three-body problem. The general solution to the three-body problem must be obtained numerically, and even with these tools, the solution is chaotic: that is small changes in initial conditions lead to widely diverging behavior.

For analysis of space missions, such as a spacecraft moving between planets, or from earth to the moon, we may apply the model known as the restricted three-body problem, in which the mass of a small spacecraft is affected by the gravitational forces from two bodies, but the large bodies do not feel the influence of the spacecraft.

The restricted three-body problem is most easily analyzed/understood in a coordinate system rotating with the two primary bodies. Since the solution of the two-body problem is that of the two primary masses rotating at constant angular velocity  $\Omega$  about their center of mass, in a coordinate system rotating with angular velocity  $\Omega$ , these two bodies are stationary. The third body moves in their gravitational fields and experiences the additional acceleration terms associated with the rotation of the coordinate system.

Consider a two-body system of mass  $M_1$  and  $M_2$  in an inertial reference frame. We add a small mass  $m_3$  which does not affect the large masses.



We consider planar x,y motion only. The equations of motion are

$$\ddot{x}_1 = -\frac{GM_2(x_1 - x_2)}{((x_1 - x_2)^2 + (y_1 - y_2)^2)^{3/2}} \quad (1)$$

$$\ddot{x}_2 = -\frac{GM_1(x_2 - x_1)}{((x_2 - x_1)^2 + (y_2 - y_1)^2)^{3/2}} \quad (2)$$

$$\ddot{x}_3 = -\frac{GM_1(x_3 - x_1)}{((x_3 - x_1)^2 + (y_3 - y_1)^2)^{3/2}} - \frac{GM_2(x_3 - x_2)}{((x_3 - x_2)^2 + (y_3 - y_2)^2)^{3/2}} \quad (3)$$

$$\ddot{y}_1 = -\frac{GM_2(y_1 - y_2)}{((x_1 - x_2)^2 + (y_1 - y_2)^2)^{3/2}} \quad (4)$$

$$\ddot{y}_2 = -\frac{GM_1(y_2 - y_1)}{((x_2 - x_1)^2 + (y_2 - y_1)^2)^{3/2}} \quad (5)$$

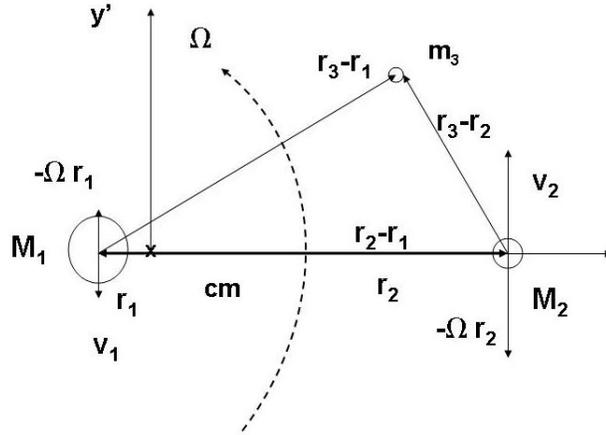
$$\ddot{y}_3 = -\frac{GM_1(y_3 - y_1)}{((x_3 - x_1)^2 + (y_3 - y_1)^2)^{3/2}} - \frac{GM_2(y_3 - y_2)}{((x_3 - x_2)^2 + (y_3 - y_2)^2)^{3/2}} \quad (6)$$

The solution to the motion of  $M_1$  and  $M_2$  is a Kepler problem, where the two bodies move in circular orbits about their center of mass with a frequency

$$\Omega = \sqrt{\frac{G(M_1 + M_2)}{R^3}}, \quad (7)$$

where  $R$  is the distance between them.  $R = -r_1 + r_2$  ( $r_1$  is negative). We can use these equations to solve numerically for the motion of the small mass  $m_3$  in the presence of the gravitational field of  $M_1$  and  $M_2$ ; in general no analytic solution is possible.

However, considerable insight can be gained about the behavior of the small mass  $m_3$  by moving into a coordinate system in which the two large masses  $M_1$  and  $M_2$  are stationary. Since the solution to the Kepler problem for these two masses is a rotation of angular velocity  $\Omega$  about the center of mass, if we place the origin of our coordinate system at the center of mass and rotate it with angular velocity  $\Omega$ , we will see that  $M_1$  and  $M_2$  are at rest; their velocities  $v_1$  and  $v_2$  which are obtained from the solution to the Kepler problem are exactly canceled by the effect of rotation, in which the velocity at  $r_1$  and  $r_2$  are canceled since  $\Omega r_2$  and  $\Omega r_1$  are subtracted from  $v_2$  and  $v_1$  as the effect of the rotating coordinate system.



However, the acceleration in inertial space  $\ddot{x}$  and  $\ddot{y}$  must be modified to account for the rotating coordinate system. From Lecture 8, equations(9), and (10) the acceleration for planar motion  $x'(t)$  and  $y'(t)$  in a rotating coordinate system is given by

$$\ddot{x}(t) = \frac{d^2 x'}{dt^2} - x' \Omega^2 - 2\Omega \frac{dy'}{dt} \quad (8)$$

$$\ddot{y}(t) = \frac{d^2 y'}{dt^2} - y' \Omega^2 + 2\Omega \frac{dx'}{dt} \quad (9)$$

$$\ddot{z}(t) = \frac{d^2 z'}{dt^2} \quad (10)$$

Here we have included the acceleration in the z (out of plane) direction since it is unmodified by the rotation of the coordinate system. This will allow us to study out of plane motions if we wish.

We now modify the equations for the motion of the three masses in their mutual gravitational field by modifying the acceleration according to the formulae given in equation (8), (9), and (10). The gravitational terms are unmodified since they are unaffected by the rotation of the coordinate system. (The gravitational force have no time behavior and are unaffected by relative motion.) However, only the motion of the small mass  $m_3$  is of interest, since by the construction of the problem,  $M_1$  and  $M_2$  are stationary in our coordinate system. We include the  $z(t)$  equation which is very straightforward;  $\ddot{z}(t)$  is unaffected by the rotation of the coordinate system; by our problem statement, the motion of the small mass does not affect the motion of the two large masses; and the motion of the two large masses is planar, so that  $z_1$  and  $z_2$  are always zero.

$$\begin{aligned} \frac{d^2 x'_3}{dt^2} - x'_3 \Omega^2 - 2\Omega \frac{dy'_3}{dt} &= -\frac{GM_1(x'_3 - x'_1)}{((x'_3 - x'_1)^2 + (y'_3 - y'_1)^2 + z_3^2)^{3/2}} - \frac{GM_2(x'_3 - x'_2)}{((x'_3 - x'_2)^2 + (y'_3 - y'_2)^2 + z_3^2)^{3/2}} \quad (11) \\ \frac{d^2 y'_3}{dt^2} - y'_3 \Omega^2 + 2\Omega \frac{dx'_3}{dt} &= -\frac{GM_1(y'_3 - y'_1)}{((x'_3 - x'_1)^2 + (y'_3 - y'_1)^2 + z_3^2)^{3/2}} - \frac{GM_2(y'_3 - y'_2)}{((x'_3 - x'_2)^2 + (y'_3 - y'_2)^2 + z_3^2)^{3/2}} \\ \frac{d^2 z'_3}{dt^2} &= -\frac{GM_1(z'_3)}{((x'_3 - x'_1)^2 + (y'_3 - y'_1)^2 + (z'_3)^2)^{3/2}} - \frac{GM_2(z'_3)}{((x'_3 - x'_2)^2 + (y'_3 - y'_2)^2 + (z'_3)^2)^{3/2}} \end{aligned}$$

We now drop the 's for convenience, concentrate on the motion of  $m_3$  in the  $x, y$  plane, and rewrite the equations **moving the centripetal acceleration to the right side**. The centripetal acceleration  $-x\Omega^2 \vec{i} -$

$y\Omega^2\vec{j}$  can be written as the gradient of a scalar:  $-x\Omega^2\vec{i} - \Omega^2\vec{j} = -\nabla(\frac{1}{2}\Omega^2r^2)$ . If we do this, we can see that the entire right side of the equation can be written as the gradient of a scalar function of  $r$ , and thus could be thought of as a potential, related to a conservative force field as  $F_c = -\nabla V(r)$ . It contains both the gravitational potential and the potential formed from the centrifugal force. Because we have left the Coriolis acceleration on the left side of the equation, we do not have a simple equation for the behavior of a particle in a conservative force field in inertial space. However, if we are interested in the spatial behavior of the force field seen by a particle at rest, this formulation gives great insights. We write the equation for the motion of  $m_3$  in the  $x, y$  plane as

$$\ddot{\vec{r}} + 2(\vec{\Omega} \times \frac{d\vec{r}}{dt}) = -\nabla(V_g(r) - \frac{1}{2}\Omega^2r^2) \quad (12)$$

where  $V_g(r)$  is the gravitational potential for the two bodies:  $M_1$  and  $M_2$  located at  $x = -r_1, y = 0$  and  $x = r_2, y = 0$ , as shown in the figures.

From Lecture 13, we have that the gravitational potential/per unit mass of  $m_3$  is

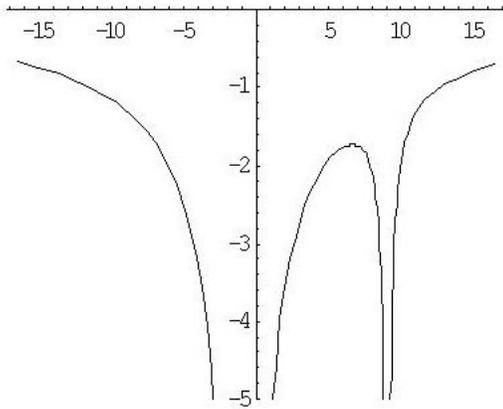
$$V_g(r) = -\frac{GM_1}{\sqrt{(x+r_1)^2+y^2}} - \frac{GM_2}{\sqrt{(x-r_2)^2+y^2}} \quad (13)$$

so that the total "potential" is

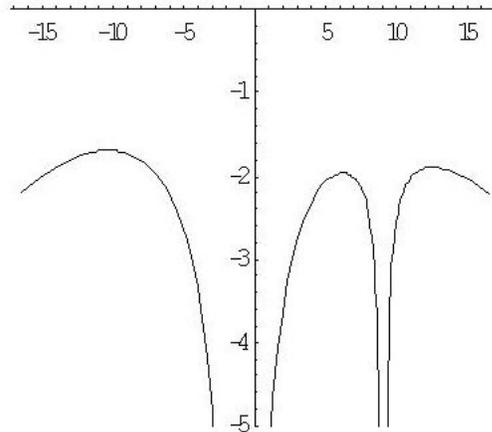
$$V(r) = -\frac{GM_1}{\sqrt{(x+r_1)^2+y^2}} - \frac{GM_2}{\sqrt{(x-r_2)^2+y^2}} - \frac{1}{2}\Omega^2r^2 \quad (14)$$

where the effect of centrifugal force has been written as a potential,  $V_\Omega = -\frac{1}{2}\Omega^2r^2$ . We shall refer to the combination of gravitational and centrifugal potential as a pseudo-potential, since when the velocity is not zero, the additional Coriolis term must be included to determine the dynamics of the particle; in this case we do not have the simple picture of a particle exchanging energy between potential and kinetic. We will use it to investigate the neighborhood, and specifically to ask whether there are any points where a particle might be placed and experience no force.

For convenience, we set  $G = 1, M_1 = 10, M_2 = 1$ , and  $R = 10$ . This gives  $r_1 = 10/11$  and  $r_2 = 100/11$  and  $\Omega = \sqrt{11/1000}$ . We first examine the gravitational potential given by equation (15) along the  $x$  axis in comparison with the pseudo-potential. The result are shown in the figure. We recall, that at any point,  $\vec{F} = -\nabla V$ . Although we are only looking at the behavior along  $y = 0$ , we see an interesting differences between the behavior of the gravitational potential and the pseudo-potential.



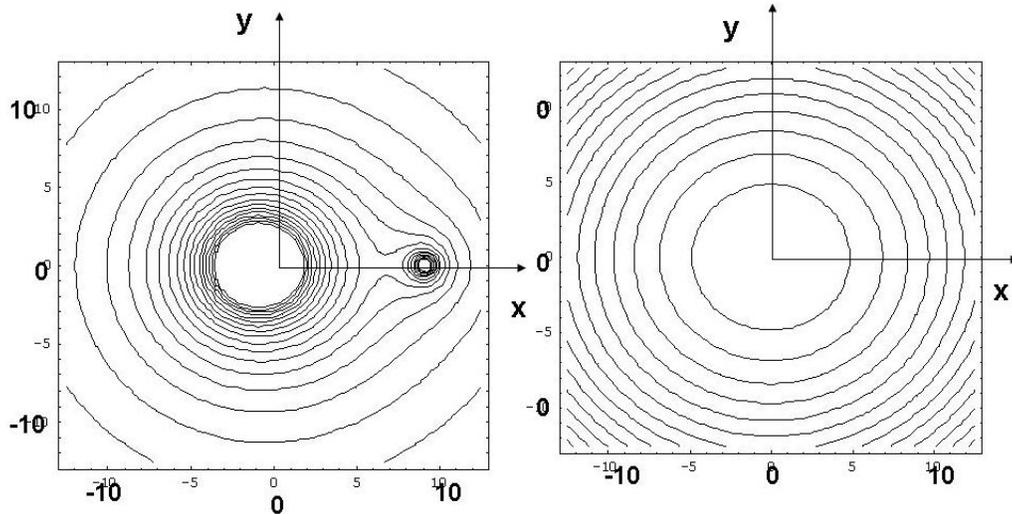
**Gravitational Potential**



**Pseudo-Potential**

For the gravity potential, we do see a point where the gravitational pull of the two bodies is equal and opposite (Think of the earth-moon system.). However, for the pseudo-potential, we see **three points** where the gravitational plus centrifugal forces balance. To explore this further, we need to move into two dimensions in the plane. The result for the gravitational potential alone is not meaningful, since we are in a rotating coordinate system. The centrifugal force is a "real" force and must be included in any discussion of force; however the gravitational potential does give some physical insight into what's going on. We now move into two dimensions and examine the contour plot of the gravitational and the centrifugal potential; lines of constant potential in the x, y plane.

**Contour Plots: lines of constant potential**



**Gravitational Potential**

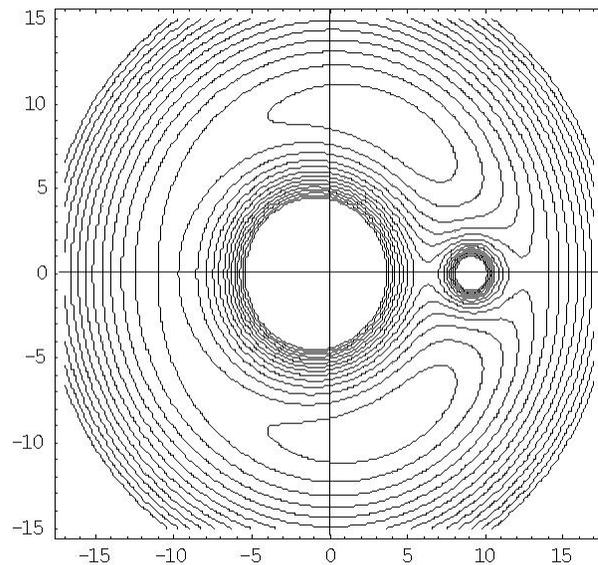
**Centrifugal-Potential**

The figure shows lines of constant potential for the gravitational potential of the two bodies,  $M_1$  and  $M_2$ . We also show the potential for the centrifugal term which is a simple  $-\frac{1}{2}\Omega^2 r^2$  term; lines of constant potential

are circles.

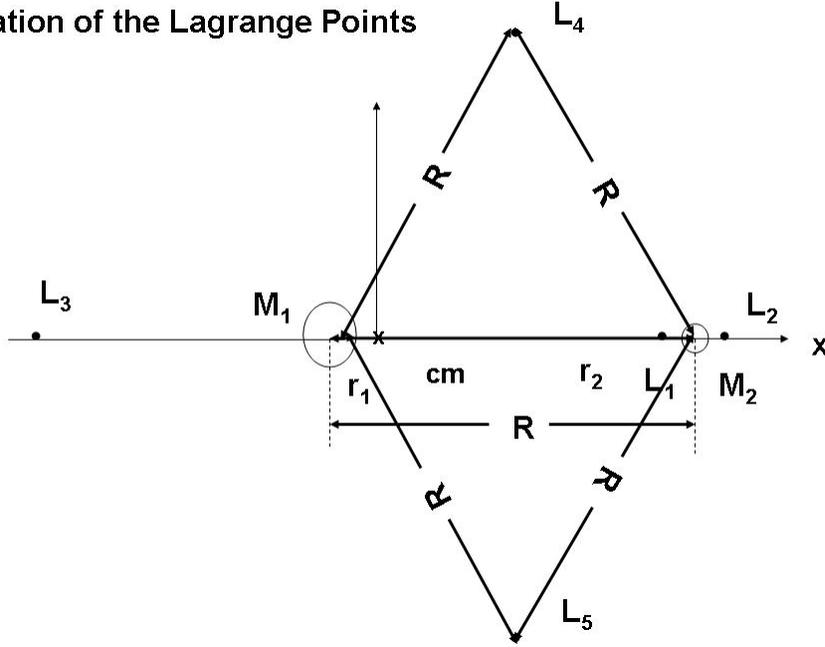
Now we combine these expressions to form the total potential for the two-body problem in a rotating coordinate system in which the two bodies are at rest. This will allow us to answer the question: for a small body  $m_3$  wandering about this neighborhood, are there any points where no force will be exerted on  $m_3$  and therefore points at which  $m_3$  might be able to remain at rest under the action of a balance of forces. There are five such points called the Lagrange points. They play an important role in the theory of planetary motion, and the design of space missions.

**Contour Plot  
of the Potential  
for the Two-Body  
problem in a  
Rotating  
Coordinate  
System**



We see several regions of this figure which bear further examination. Recall from our examination of the behavior of the pseudo-potential along the  $x$  axis, we identify three possible points where no force in the  $x$  direction would be exerted on a particle. In fact, due to the symmetry of the potential in the  $y$  direction, no  $y$  force will be felt on a particle located on the  $x$  axis. Therefore these three points are equilibrium points, in the sense that no force acts upon a particle at these points. Recall that following any discussion of equilibrium, is a discussion of stability: a ball balanced upon an overturned bowl is in equilibrium but not stable; a ball at the bottom and inside a bowl is at equilibrium and stable. These three points are visible in the contour plot, although the point outboard of  $M_1$  is indistinct; it is there. Before completing our discussion, we turn briefly to the closed contours off of the  $x$  axis. If this contour contains a "bowl" or a potential well, we have a possible point of equilibrium. For a closed contour of a continuous function, we are guaranteed a point of zero slope somewhere within the contour, a possible point of equilibrium; however, our dynamics includes Coriolis forces which could affect the stability. We now analyze the location of these points in more detail and greater generality.

## Location of the Lagrange Points



We now return to the two-body problem and do not choose the particular numerical values for  $M_1$ ,  $M_2$  and  $R$ . We do take  $G = 1$  as before (you can think of this a particular non-dimensionalization of the equations; in any case from the form of  $V_g$  and the form of  $\Omega$ ,  $G$  drops out. We introduce

$$\vec{r} = x(t)\vec{i} + y(t)\vec{j} \quad (15)$$

$$r_1 = -\alpha R \quad (16)$$

$$r_2 = \beta R \quad (17)$$

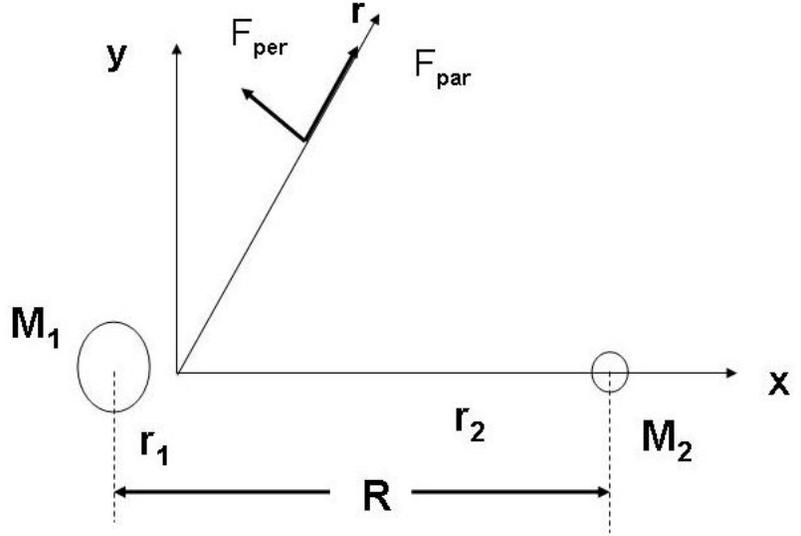
where  $\alpha = \frac{M_2}{M_1+M_2}$ ;  $\beta = \frac{M_1}{M_1+M_2}$  and  $\Omega = \sqrt{\frac{G(M_1+M_2)}{R^3}}$  as required by the solution of the two-body problem. With this formulation, we seek solutions for the points  $x$  and  $y$  where the force on a particle would be zero. This is a complex problem, involving finding the roots of three fifth-order polynomials. We could do such a solution numerically, but for greater insight we seek an analytic solution. Such a solution can easily be found for small  $\alpha$ , where  $\alpha$  is the mass ratio. In fact the existence of such points requires that  $\alpha$  be small, so this is not a significant restriction on the utility of the result. Expanding the total  $x$  and  $y$  forces on the particle for small  $\alpha$  and setting the result equal to zero gives the following approximations for the location of these three equilibrium points along the  $x$  axis. They are called Lagrange points  $L_1$ ,  $L_2$  and  $L_3$ .

$$L_1 = R \left( 1 - \left( \frac{\alpha}{3} \right)^{1/3} \right) \quad (18)$$

$$L_2 = R \left( 1 + \left( \frac{\alpha}{3} \right)^{1/3} \right) \quad (19)$$

$$L_3 = -R \left( 1 + \left( \frac{5\alpha}{12} \right) \right) \quad (20)$$

Identifying the remaining Lagrange point requires that we find a force balance for  $y \neq 0$ .



Referring to the figure, we note that since the centrifugal force is aligned with vector  $\vec{r}$ , a force balance—if indeed there is one—would involve only gravitational terms. Thus in a direction perpendicular to  $\vec{r}$

$$F_{per} = \alpha\beta\Omega^2 R^3 \left( \frac{1}{\sqrt{(x - R\beta)^2 + y^2}^{3/2}} - \frac{1}{\sqrt{(x + R\alpha)^2 + y^2}^{3/2}} \right) = 0 \quad (21)$$

resulting in a solution  $x = \frac{1}{2}(\beta R - \alpha R)$ : that is the  $x$  location of the Lagrange point—if there is one, lies exactly half way between the two masses. With the result, the equation for the forces parallel to  $\vec{r}$  reduce to

$$F_{par} = \Omega^2 \frac{(x^2 + y^2)}{R} \left( \frac{1}{R^3} - \frac{1}{(x - R\beta)^2 + y^2} \right) \quad (22)$$

Since this result equates the distance between the point in question to the magnitude of  $R$ , we conclude that the Lagrange point is at a distance  $R$  from  $M_2$  and by symmetry from  $M_1$ :  $L_4$  is at positive  $y$ ;  $L_5$  its mirror image. Since the masses are  $R$  apart on the  $x$  axis, the location of  $L_4$  is at the peak of an equilateral triangle:  $R$  on all three sides. We add these points to the figure showing location of the Lagrange points. This is a remarkable result. The location of the  $L_4$  and  $L_5$ , which have both  $x$  and  $y$  positions are

$$L_4 = \left( \frac{R(M_1 - M_2)}{2(M_1 + M_2)}, \frac{\sqrt{3}}{2}R \right) \quad (23)$$

$$L_5 = \left( \frac{R(M_1 - M_2)}{2(M_1 + M_2)}, -\frac{\sqrt{3}}{2}R \right) \quad (24)$$

## Stability and Application of the Lagrange Points

So much can be said about these results. In examining the stability of the mass placed at these equilibrium points, done as a note later in the section, we find that  $L_4$  and  $L_5$  are stable—due in part to the Coriolis force which acts on a particle in motion—; and  $L_1$ ,  $L_2$ , and  $L_3$  are unstable. However, even though small perturbations to a mass located at these Lagrange points would produce an unstable motion, the amplification

rate is small and it is far easier to locate a spacecraft at these points and apply an automatic control system for "station keeping" than to locate a spacecraft at rest at an arbitrary point in space. So, even though these point are points of unstable equilibrium, it requires only a small expenditure of fuel to keep spacecraft at these points. Current space missions place objects in orbits about these Lagrange points for various purposes, achieving very practical results.

Many space missions have used these points. For example the current plan is to locate the James Webb optical telescope in an orbit about the  $L_{earth-sun_2}$  point where it will be shielded from direct sunlight by the earth, and can be kept on station in a efficient manner.

In the natural world, groups of asteroids –called the Trojan asteroids–have been observed to cluster at the Sun-Jupiter  $L_4$  and  $L_5$  Lagrange points. Trojan asteroids have also been found at the  $L_4$  and  $L_5$  Lagrange point of Mars and Neptune.

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**Note**

***Stability Analysis of the Lagrange Points***

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We study the stability of the various Lagrange points by expanding the equations of motion about each equilibrium position,  $L_1$  to  $L_5$ . Setting

$$x = Lx_i + x_p \tag{25}$$

$$y = Ly_i + y_p \tag{26}$$

$$z = z_p \tag{27}$$

where  $Lx_i$  and  $Ly_i$  are the  $x$  and  $y$  locations of the  $i$ 'th Lagrange (equilibrium) points and  $x_p$ ,  $y_p$ , and  $z_p$  are small perturbations about these locations. We can for a very small price, include the  $z$  direction, since the Lagrange points are located in the  $x, y$  plane, for which  $Lz_i = 0$ . In studying the stability, we are essentially deriving the equations for small "oscillations" about the equilibrium point; we intend to linearize these equations. Oscillatory or decaying solutions imply stability; exponentially diverging solutions imply instability. To obtain the equations governing small motions from equilibrium, we expand equation (12) for small displacements about the various Lagrange points. Since the gradient of the potential is zero at these equilibrium points, equation (12) in component form becomes

$$\frac{d^2 x_p}{dt^2} - 2\Omega \frac{dy_p}{dt} = -V_{xx}x_p - V_{xy}y_p - V_{xz}z_p \tag{28}$$

$$\frac{d^2 y_p}{dt^2} + 2\Omega \frac{dx_p}{dt} = -V_{xx}x_p - V_{xy}y_p - V_{xz}z_p \tag{29}$$

$$\frac{d^2 z_p}{dt^2} = -V_{xx}x_p - V_{xy}y_p - V_{xz}z_p \tag{30}$$

The various second derivatives of the pseudo potential  $-V(x, y, z)$  are given in the table for the various Lagrange points. Many of them are zero. This derivation allows the construction of the differential equations governing small displacement motion about the Lagrange points. From this we find that  $L_1$ ,  $L_2$  and  $L_3$  are unstable, and  $L_4$  and  $L_5$  are stable. These equations provide the foundation for the analysis of station

Table 1: Stability Coefficients for the Lagrange Points

point	$-VXX$	$-VYY$	$-VZZ$	$-VXY$	$-VXZ$	$-VYZ$
L1	$9\Omega^2$	$-3\Omega^2$	$-4\Omega^2$	0	0	
L2	$9\Omega^2$	$-3\Omega^2$	$-4\Omega^2$	0	0	
L3	$3\Omega^2$	$-\frac{7}{8}\alpha\Omega^2$	$-\Omega^2$	0	0	
L4	$-\frac{3}{4}\Omega^2$	$-\frac{9}{4}\Omega^2$	$-\Omega^2$	$-\frac{3\sqrt{3}}{4}\Omega^2$	0	
L5	$-\frac{3}{4}\Omega^2$	$-\frac{9}{4}\Omega^2$	$-\Omega^2$	$-\frac{3\sqrt{3}}{4}\Omega^2$	0	

keeping of a spacecraft at a Lagrange point. Thus, as previously mentioned, even though the motions of a spacecraft located at one of the various co-linear Lagrange point would be unstable, the rate of amplification is small and the Lagrange points are very attractive locations for various space missions.

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