

## Lecture L11 - Conservation Laws for Systems of Particles

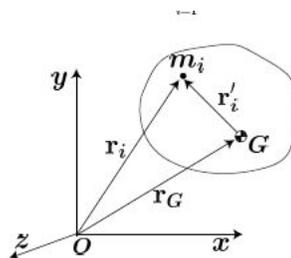
In this lecture, we will revisit the application of Newton's second law to a system of particles and derive some useful relationships expressing the conservation of angular momentum. We also specialize these results to two-dimensional rigid bodies.

### Center of Mass

Consider a system made up of  $n$  particles. A typical particle,  $i$ , has mass  $m_i$ , and, at the instant considered, occupies the position  $\mathbf{r}_i$  relative to a frame  $xyz$ . We can then define the center of mass,  $G$ , as the point whose position vector,  $\mathbf{r}_G$ , is such that,

$$\mathbf{r}_G = \frac{1}{m} \left( \sum_{i=1}^n m_i \mathbf{r}_i \right) . \quad (1)$$

Here,  $m$  is the total mass of the system given by  $m = \sum_{i=1}^n m_i$ .



It is important to note that the center of mass is a property of the system and does not depend on the reference frame used. In particular, if we change the location of the origin  $O$ ,  $\mathbf{r}_G$  will change, but the absolute position of the point  $G$  within the system will not. Often, it will be convenient to describe the motion of particle  $i$  as the motion of  $G$  plus the motion of  $i$  relative to  $G$ . To this end, we introduce the relative position vector,  $\mathbf{r}'_i$ , and write,

$$\mathbf{r}_i = \mathbf{r}_G + \mathbf{r}'_i . \quad (2)$$

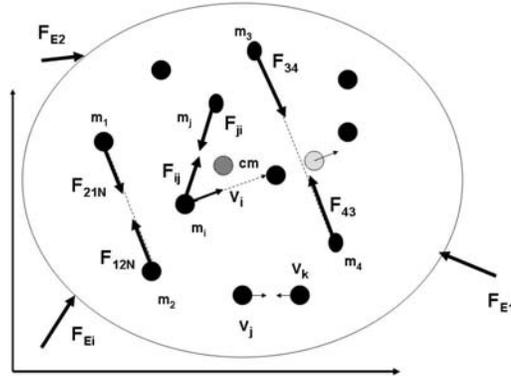
It follows immediately, from the definition of the center of mass (1) and the definition of the relative vector  $\mathbf{r}'_i$  (2), that,

$$\sum_{i=1}^n m_i \mathbf{r}'_i = \sum_{i=1}^n m_i (\mathbf{r}_i - \mathbf{r}_G) = \sum_{i=1}^n m_i \mathbf{r}_i - m \mathbf{r}_G = 0 . \quad (3)$$

This result will simplify our later analysis.

# Forces

In order to derive conservation laws for our system, we isolate it a little more carefully, identify what mass particles it contains and what forces act upon the individual particles.



We will consider two types of forces acting on the particles :

**External** forces arising outside the system. We will denote the resultant of all the **external** forces acting on the system as  $\sum_{i=1}^N \mathbf{F}_{Ei} = \mathbf{F}$ .

**Internal** forces due to pairwise particle interactions. Let  $\mathbf{f}_{ij}$  denote the force that particle  $j$  exerts on particle  $i$  directed along the line joining the two particles  $i$  and  $j$ . This force could arise from gravitation attraction or from internal force due to the connections between particles. It could also arise from collisions between individual particles that, as we have seen, produce equal and opposite impulsive forces that conserve momentum.

By Newton's third law, these internal forces act in pairwise equal and opposite directions. Therefore,  $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$ , where  $\mathbf{f}_{ji}$  is the force that particle  $i$  exerts on particle  $j$  along the line joining the particles. The total internal force on particle  $i$  is then

$$\sum_{j=1, j \neq i}^n \mathbf{f}_{ij},$$

and, if we sum over all particles, we have

$$\sum_{i=1}^n \sum_{j=1, j \neq i}^n \mathbf{f}_{ij} = \mathbf{0},$$

since, for every force,  $\mathbf{f}_{ij}$ , there is an equal and opposite force,  $\mathbf{f}_{ji}$  for both normal and tangential forces. (The notation  $j \neq i$  means to exclude  $j=i$  in the sum since of course the particle induces no net force upon itself.)

## Conservation of Linear Momentum

The linear momentum of the system is defined as,

$$\mathbf{L} = \sum_{i=1}^n m_i \mathbf{v}_i . \quad (4)$$

From equation (2), we have that  $\mathbf{v}_i = \dot{\mathbf{r}}_i = \dot{\mathbf{r}}_G + \dot{\mathbf{r}}'_i$ , which, combined with the above equation, gives,

$$\mathbf{L} = \sum_{i=1}^n m_i (\dot{\mathbf{r}}_G + \dot{\mathbf{r}}'_i) = \sum_{i=1}^n m_i \mathbf{v}_G + \frac{d}{dt} \left( \sum_{i=1}^n m_i \mathbf{r}'_i \right) = m \mathbf{v}_G , \quad (5)$$

since by the definition of center of mass  $\sum_{i=1}^n m_i \mathbf{r}'_i = \mathbf{0}$ . We now consider the time variation of the linear momentum. If we assume that the reference frame  $xyz$  is inertial, then, starting from equation (4), we have,

$$\dot{\mathbf{L}} = \sum_{i=1}^n m_i \mathbf{a}_i = \sum_{i=1}^n (\mathbf{F}_i + \sum_{j=1, j \neq i}^n \mathbf{f}_{ij}) = \sum_{i=1}^n \mathbf{F}_i = \mathbf{F} , \quad (6)$$

where  $\mathbf{F}$  is the sum of all external forces acting on the system. Since the sum of the internal forces balance when summed over  $i$  and  $j$ , we are left with only the summation over the external forces. Thus, for a system of particles, we have that,

$$\dot{\mathbf{L}} = \mathbf{F} . \quad (7)$$

Note that, from equation (5), we can also write  $\dot{\mathbf{L}} = m \mathbf{a}_G$ . This is a powerful result. Note that the center of mass is in general not fixed to a particular particle but is a point in space about which the individual particles move.

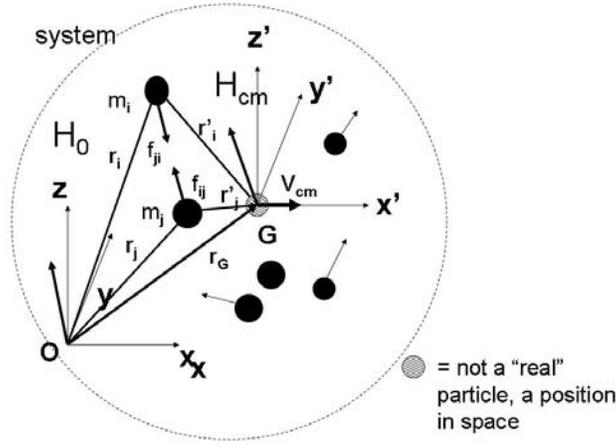
These ideas also describe the conservation of linear momentum under external and internal collisions. Since individual internal collisions between particles in the system conserve momentum, the sum of their interactions also conserves momentum. If we consider an external particle imparting momentum to the system, it could be treated as an external impulse. Conversely we can consider the particle about to collide to be a part of the system, and include its momentum as part of total system momentum, which is then conserved by Newton's law.

## Conservation of Angular Momentum

Since the angular momentum is defined with respect to a point in space, we will consider two cases, using a different reference point for each case: 1) conservation of angular momentum about a fixed (or more generally a non-accelerating) point  $O$ ; and 2) conservation of angular momentum about the center of mass, which could be accelerating in inertial space.

The figure shows the system: the coordinate system  $x, y, z$  about the fixed point  $O$ , the coordinate system  $x', y', z'$  about the center of mass,  $G$ , the internal forces between particles, the angular momentum  $\mathbf{H}_O$  about the point  $O$  and the angular momentum  $\mathbf{H}_G$  about the center of mass, point  $G$ , which in general are not equal.

## Conservation of Angular Momentum about a Fixed Point $O$



The angular momentum of a system of particles about a fixed point,  $O$ , is the sum of the angular momentum of the individual particles,

$$\mathbf{H}_O = \sum_{i=1}^n (\mathbf{r}_i \times m_i \mathbf{v}_i) . \quad (8)$$

The time variation of  $\mathbf{H}_O$  can be written as,

$$\dot{\mathbf{H}}_O = \sum_{i=1}^n (\dot{\mathbf{r}}_i \times m_i \mathbf{v}_i) + \sum_{i=1}^n (\mathbf{r}_i \times m_i \dot{\mathbf{v}}_i) = \mathbf{0} + \sum_{i=1}^n (\mathbf{r}_i \times (\mathbf{F}_i + \sum_{j=1, j \neq i}^n \mathbf{f}_{ij})) = \sum_{i=1}^n (\mathbf{r}_i \times \mathbf{F}_i) + \sum_{i=1}^n M_i . \quad (9)$$

where we replace  $m\dot{\mathbf{v}}_i$  by the sum of the forces acting on particle  $i$ :  $m\dot{\mathbf{v}}_i = (\mathbf{F}_i + \sum_{j=1, j \neq i}^n \mathbf{f}_{ij})$ .  $\sum_{i=1}^n M_i$  is the sum of any external moments that act on the system. The term  $(\dot{\mathbf{r}}_i \times m_i \mathbf{v}_i)$  in equation (9) is zero; since  $\dot{\mathbf{r}}_i = \mathbf{v}_i$ , the two vector are parallel and their cross-product is zero. In the second term we may write  $\mathbf{r}_i \times \mathbf{f}_{ij} + \mathbf{r}_j \times \mathbf{f}_{ji} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{f}_{ij} = \mathbf{0}$  since the forces are aligned with  $(\mathbf{r}_i - \mathbf{r}_j)$ , and their values are equal and opposite, their cross product with  $(\mathbf{r}_i - \mathbf{r}_j)$  is zero, and therefore, the internal forces have no net effect on the total angular momentum change of the particle system. Therefore

$$\dot{\mathbf{H}}_O = \sum_{i=1}^n (\mathbf{r}_i \times m_i \dot{\mathbf{v}}_i) = \sum_{i=1}^n (\mathbf{r}_i \times \mathbf{F}_i) + \sum_{i=1}^n M_i . \quad (10)$$

When evaluating the moments which act to change the angular momentum from equation (13), we see that the sum of the internal moments is zero so that the only moment which acts to change the angular momentum is the moments created by the external forces about the point  $O$  plus any external moments applied to the system.

Thus, we have that

$$\dot{\mathbf{H}}_O = \mathbf{M}_O , \quad (11)$$

where  $\mathbf{M}_O = \sum_{i=1}^n (\mathbf{r}_i \times \mathbf{F}_i) + \sum_{i=1}^n M_i$  is the total moment, about  $O$ , due to the applied external forces plus any external moments.

## Conservation of Angular Momentum about $G$

The angular momentum about the center of mass  $G$  is given by,

$$\mathbf{H}_G = \sum_{i=1}^n (\mathbf{r}'_i \times m_i \mathbf{v}_i) . \quad (12)$$

Taking the time derivative of equation (2), we obtain

$$\mathbf{v}_i = \dot{\mathbf{r}}_i = \dot{\mathbf{r}}_G + \dot{\mathbf{r}}'_i = \mathbf{v}_G + \mathbf{v}'_i . \quad (13)$$

Inserting this expression into equation 12, we obtain

$$\mathbf{H}_G = \sum_{i=1}^n (\mathbf{r}'_i \times m_i (\dot{\mathbf{r}}_G + \dot{\mathbf{r}}'_i)) = \sum_{i=1}^n (\mathbf{r}'_i \times m_i \dot{\mathbf{r}}_G) + \sum_{i=1}^n (\mathbf{r}'_i \times m_i \dot{\mathbf{r}}'_i) = \sum_{i=1}^n (\mathbf{r}'_i \times m_i \mathbf{v}'_i) , \quad (14)$$

since  $\sum_{i=1}^n (\mathbf{r}'_i \times m_i \dot{\mathbf{r}}_G) = -\dot{\mathbf{r}}_G \times \sum_{i=1}^n m_i \mathbf{r}'_i = \mathbf{0}$  (see equation 3). We note that equations (12) and (14) give us alternative representations for  $\mathbf{H}_G$ . Equation (12) is called the *absolute* angular momentum (since it involves absolute velocities,  $\mathbf{v}_i$ ), whereas equation (14) is called the *relative* angular momentum (since it involves velocities,  $\mathbf{v}'_i$ , relative to  $G$ ). When  $G$  is chosen to be the origin for the relative velocities, both the absolute and relative angular momentum are identical. In general, the absolute and relative angular momentum with respect to an *arbitrary point* are not the same.

We can now go back to equation (12) and consider the time variation of  $\mathbf{H}_G$ ,

$$\dot{\mathbf{H}}_G = \sum_{i=1}^n (\dot{\mathbf{r}}'_i \times m_i (\mathbf{v}_G + \dot{\mathbf{r}}'_i)) + \sum_{i=1}^n (\mathbf{r}'_i \times m_i \dot{\mathbf{v}}'_i) = \mathbf{0} + \sum_{i=1}^n (\mathbf{r}'_i \times \mathbf{F}_i) + \sum_{i=1}^n M_i . \quad (15)$$

In the above equation, the term  $\dot{\mathbf{r}}'_i \times m_i \dot{\mathbf{r}}'_i$  is clearly zero, and  $\sum_{i=1}^n (\mathbf{r}'_i \times m_i \mathbf{v}_G) = -\mathbf{v}_G \times \sum_{i=1}^n m_i \mathbf{r}'_i = -\mathbf{v}_G \times d(\sum_{i=1}^n m_i \mathbf{r}'_i)/dt = \mathbf{0}$ . Thus, we have that

$$\dot{\mathbf{H}}_G = \sum_{i=1}^n (\mathbf{r}'_i \times m_i \dot{\mathbf{v}}'_i) = \mathbf{M}_G . \quad (16)$$

Here,  $\mathbf{M}_G = \sum_{i=1}^n (\mathbf{r}'_i \times \mathbf{F}_i) + \sum_{i=1}^n M_i$ , is the total moment, about  $G$ , of the applied external forces plus any external moments. Note that external forces in general produce unequal moments about  $O$  and  $G$  while applied external moments (torques) produce the same moment about  $O$  and  $G$ .

The above expression is very powerful and allows us to solve, with great simplicity, a large class of problems in rigid body dynamics. Its power lies in the fact that it is applicable in very general situations: In the derivation of equation (16), we have made no assumptions about the motion of the center of mass,  $G$ . That is, equation (16) is valid even when  $G$  is *accelerated*. We have implicitly assumed that the reference frame used to describe  $\mathbf{r}'_i$  in equation 13 is non-rotating with respect to the fixed frame  $xyz$  (otherwise, we would have written  $\dot{\mathbf{r}}'_i = \mathbf{v}'_i + \boldsymbol{\omega}' \times \mathbf{r}'_i$ , with  $\boldsymbol{\omega}'$ , the angular velocity of the frame considered). It is not difficult to show that equation (16) is still valid if the reference frame rotates, provided the angular velocity is *constant*. If the reference frame rotates with a constant angular velocity, the angular momentum will differ from that of equations (12) and (14) by a constant, but equation 16 still will be valid.

Finally, by combining equations 30 and 12, the angular momentum about a fixed point,  $O$ , can be expressed as a function of the angular momentum about the center of mass, as,

$$\mathbf{H}_O = \mathbf{r}_G \times m\mathbf{v}_G + \mathbf{H}_G. \quad (17)$$

Just as we could incorporate collisions in our statement of conservation of linear momentum, we can incorporate collision in our statement of conservation of angular momentum. Collisions conserve both linear and angular momentum. Just as changes in linear momentum result for linear impulses, changes in angular momentum result from angular impulses.

## Kinetic Energy for Systems of Particles

Here, we derive the expression for the kinetic energy of a system of particles that will be used in the following lectures. A typical particle,  $i$ , will have a mass  $m_i$ , an absolute velocity  $\mathbf{v}_i$ , and a kinetic energy  $T_i = (1/2)m_i\mathbf{v}_i \cdot \mathbf{v}_i = (1/2)m_iv_i^2$ .

The total kinetic energy of the system,  $T$ , is simply the sum of the kinetic energies for each particle,

$$T = \sum_{i=1}^n T_i = \sum_{i=1}^n \frac{1}{2}m_iv_i^2.$$

It is convenient to decompose the velocity of each particle,  $\mathbf{v}_i$ , into the velocity of the center of mass,  $\mathbf{v}_G$ , and the velocity relative to the center of mass,  $\dot{\mathbf{r}}'_i$ . Then,

$$T = \sum_{i=1}^n \frac{1}{2}m_i(\mathbf{v}_G + \dot{\mathbf{r}}'_i) \cdot (\mathbf{v}_G + \dot{\mathbf{r}}'_i) = \sum_{i=1}^n \frac{1}{2}m_i(v_G^2 + 2\mathbf{v}_G \cdot \dot{\mathbf{r}}'_i + \dot{r}'_i{}^2) = \frac{1}{2}mv_G^2 + \sum_{i=1}^n \frac{1}{2}m_i\dot{r}'_i{}^2,$$

since  $\mathbf{v}_G \cdot \sum_{i=1}^n m_i\dot{\mathbf{r}}'_i = 0$ , and  $\sum_{i=1}^n m_i = m$ . Thus, we see that the kinetic energy of a system of particles equals the kinetic energy of a particle of mass total  $m$  moving with the velocity of the center of mass, plus the kinetic energy due to the motion of the particles relative to the center of mass,  $G$ . We have said nothing about the conservation of energy for a system of particles. As we shall see, that depends upon the details of internal interactions and the work done by the external forces.

We will now particularize the conservation principles presented in the previous lectures to the case in which the system of particles considered is a 2D rigid body.

## Conservation of Angular Momentum for 2D Rigid Body

In subsequent Lectures, we will apply conservation of angular momentum for a general system in three dimensions. However, the dynamics of a 2D rigid body can easily be incorporated into our study of particle motion. We therefore specialize these results to the case of a 2D rigid body. The equations describing the general motion of a rigid body follow from the conservation laws for systems of particles. Since the general motion of a **2D rigid body** can be determined by three parameters (e.g.  $x$  and  $y$  coordinates of position,

and a rotation angle  $\theta$ ), we will need to supply three equations. Conservation of linear momentum yields one vector equation, or two scalar equations. The additional condition is conservation of angular momentum. We saw that there are several ways to express conservation of angular momentum. In principle, they are all equivalent, but, depending on the problem situation, the use of a particular form may greatly simplify the problem. The best choices for the origin of coordinates are: 1) the center of mass  $G$ ; 2) a fixed point  $O$ .

## Conservation of Angular Momentum about the Center of Mass

When considering a 2D rigid body, the velocity of any point relative to  $G$  consists of a pure rotation and, therefore, can be expressed as

$$\mathbf{v}'_i = \boldsymbol{\omega} \times \mathbf{r}'_i,$$

where  $\boldsymbol{\omega}$  is the angular velocity vector perpendicular to the plane of motion. Both  $\mathbf{v}'_i$  and  $\mathbf{r}'_i$  are in the plane of the motion. These two equations can be combined to give,

$$\mathbf{H}_G = \sum_{i=1}^n (\mathbf{r}'_i \times m_i (\boldsymbol{\omega} \times \mathbf{r}'_i)) = \boldsymbol{\omega} \sum_{i=1}^n m_i r_i'^2 = \boldsymbol{\omega} \int_m r'^2 dm \quad (18)$$

For a continuous body, the sum over the mass points is replaced by an integral.  $\int_m r'^2 dm$  is defined as the mass moment of inertia  $I_G$  about the center of mass. Here, we have used the vector identity,  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$ , and imposed the fact that  $\mathbf{r}'_i$  and  $\boldsymbol{\omega}$  are perpendicular for 2D planar bodies. Thus, for a 2-D rigid body, the conservation law for angular momentum about the center of mass,  $G$  is

$$I_G \boldsymbol{\alpha} = \mathbf{M}_G, \quad (19)$$

where  $I_G = \int_m r'^2 dm$ ,  $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$  and  $\mathbf{M}_G$  is the total moment about  $G$  due to external forces and external moments. Although equation (7) is a vector equation,  $\boldsymbol{\alpha}$  and  $\mathbf{M}_G$  are always perpendicular to the plane of motion, and, therefore, equation (7) only yields one scalar equation. The moment of inertia,  $I_G$ , can be interpreted as a measure of the body's resistance to changing its angular velocity as a result of applied external moments. The moment of inertia,  $I_G$ , is a scalar quantity. It is a property of the solid which indicates the way in which the mass of the solid is distributed relative to the center of mass. For example, if most of the mass is far away from the center of mass,  $r'_i$  will be large, resulting in a large moment of inertia. The dimensions of the moment of inertia are  $[M][L^2]$ .

## Conservation of Angular Momentum about a fixed point $O$

If the fixed point  $O$  is chosen as the origin, a similar result is obtained. Since for a 2D rigid body the velocity in the coordinate system fixed at the point  $O$  is

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i,$$

conservation of angular momentum gives

$$\mathbf{H}_O = \sum_{i=1}^n (\mathbf{r}_i \times m_i (\boldsymbol{\omega} \times \mathbf{r}_i)) = \boldsymbol{\omega} \sum_{i=1}^n m_i r_i^2 = \boldsymbol{\omega} r_i^2 = \boldsymbol{\omega} \int_m r^2 dm = \boldsymbol{\omega} I_O \quad (20)$$

where  $I_O$  is the moment of inertia about the point  $O$ . In general,  $I_O \neq I_G$ . The conservation law for angular momentum about the fixed point  $O$  is then

$$I_O \boldsymbol{\alpha} = \mathbf{M}_O, \quad (21)$$

where  $I_O = \int_m r^2 dm$ ,  $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$  and  $\mathbf{M}_O$  is the total applied moment due to external forces and moments (torques). Also it is important to point out that both the angular velocity  $\boldsymbol{\omega}$  and the angular acceleration  $\boldsymbol{\alpha}$  are the same for any point on a rigid body:  $\boldsymbol{\omega}_G = \boldsymbol{\omega}_O$ ,  $\boldsymbol{\alpha}_G = \boldsymbol{\alpha}_O$ .

Most textbooks on dynamics have tables of moments of inertia for various common shapes: cylinders, bars, plates. See Meriam and Kraige, *Engineering Mechanics*, DYNAMICS (Appendix B) for more examples.

## Radius of Gyration

It is common to report the moment of inertia of a rigid body in terms of the radius of gyration,  $k$ . This is defined as

$$k = \sqrt{\frac{I}{m}},$$

and can be interpreted as the root-mean-square of the mass element distances from the axis of rotation.

Since the moment of inertia depends upon the choice of axis, the radius of gyration also depends upon the choice of axis. Thus we write

$$k_G = \sqrt{\frac{I_G}{m}},$$

for the radius of gyration about the center of mass, and

$$k_O = \sqrt{\frac{I_O}{m}},$$

for the radius of gyration about the fixed point  $O$ .

## Parallel Axis Theorem

We will often need to find the moment of inertia with respect to a point other than the center of mass. For instance, the moment of inertia with respect to a given point,  $O$ , is defined as

$$I_O = \int_m r^2 dm .$$

Assuming that  $O$  is a fixed point,  $\mathbf{H}_O = I_O \boldsymbol{\omega}$ . If we know  $I_G$ , then the moment of inertia with respect to point  $O$ , can be computed easily using the parallel axis theorem. Given the relations  $r^2 = \mathbf{r} \cdot \mathbf{r}$  and  $\mathbf{r} = \mathbf{r}_G + \mathbf{r}'$ , we can then write,

$$I_O = \int_m r^2 dm = \int_m (r_G^2 + 2\mathbf{r}_G \cdot \mathbf{r}' + r'^2) dm = r_G^2 \int_m dm + 2\mathbf{r}_G \cdot \int_m \mathbf{r}' dm + \int_m r'^2 dm = m r_G^2 + I_G ,$$

since  $\int_m \mathbf{r}' dm = \mathbf{0}$ .

From this expression, it also follows that the moment of inertia with respect to an arbitrary point is **minimum** when the point coincides with  $G$ . Hence, the **minimum** value for the moment of inertia is  $I_G$ .

## Equations of Motion for a 2D rigid Body.

Now that we have developed the equation governing conservation of angular momentum for a 2D rigid body in planar motion, we can state the governing equations for this three degree of freedom system.

The conservation of linear momentum yields the vector equation,

$$m\mathbf{a}_G = \mathbf{F} , \quad (22)$$

where  $m$  is the body mass,  $\mathbf{a}_G$  is the acceleration of the center of mass, and  $\mathbf{F}$  is the sum of the external forces acting on the body.

Conservation of angular momentum requires

$$\dot{\mathbf{H}}_G = \mathbf{M}_G = \dot{\omega}I_G = \boldsymbol{\alpha}I_G \quad (23)$$

where  $I_G$  is the moment of inertia about the center of mass;  $\boldsymbol{\omega}$  is the angular velocity, whose vector direction is perpendicular to the  $x, y$  coordinate system; and  $\boldsymbol{\alpha}$  is the angular acceleration. For impulsive forces, we write that the change in angular momentum is equal to the time integral of the applied moments, whether or not these moments are impulsive (but we have to be able to do the time integral, so impulsive forces are easier).

A body fixed at a point  $O$  is a single degree of freedom system. Therefore, only one equation is required, conservation of angular momentum about the point  $O$ .

$$\dot{\mathbf{H}}_O = \mathbf{M}_O = \dot{\omega}I_O = \boldsymbol{\alpha}I_O \quad (24)$$

## Impulsive Motion

Just as we applied the continuous form of Newton's law for a particle to the motion resulting from impulsive forces and collisions, we can extend these result to impulsive linear and angular motion resulting from impulsive forces and moments.

$$\mathbf{H}_{G2} - \mathbf{H}_{G1} = \int_{t_1}^{t_2} \sum_{i=1}^n (\mathbf{r}'_i \times \mathbf{F}_i) . \quad (25)$$

where  $\int_{t_1}^{t_2} (\sum_{i=1}^n \mathbf{r}'_i \times \mathbf{F}_i) = \mathbf{A}_I$  is the **angular impulse**, the integral over time of the sum of the moments acting about the center of mass. A similar result holds for moments taken about the point  $O$ .

$$\mathbf{H}_{O2} - \mathbf{H}_{O1} = \int_{t_1}^{t_2} \sum_{i=1}^n (\mathbf{r}_i \times \mathbf{F}_i) . \quad (26)$$

where  $\int_{t_1}^{t_2} (\sum_{i=1}^n \mathbf{r}_i \times \mathbf{F}_i) = \mathbf{A}_I$

## Kinetic Energy for a 2D Rigid Body

We specialize the result of the analysis of kinetic energy for a system of particles about the center of mass for the case of a two-dimensional rigid body for which the velocity of a particle is given by

$$\mathbf{v}'_i = \boldsymbol{\omega} \times \mathbf{r}'_i . \quad (27)$$

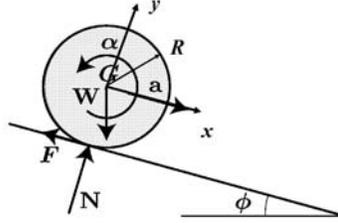
Applying this to the general form for from Equation

$$T = \sum_{i=1}^n \frac{1}{2} m_i (\mathbf{v}_G + \dot{\mathbf{r}}'_i) \cdot (\mathbf{v}_G + \dot{\mathbf{r}}'_i) = \frac{1}{2} m v_G^2 + \frac{1}{2} \boldsymbol{\omega} \int_m r'^2 dm = \frac{1}{2} m v_G^2 + \frac{1}{2} \boldsymbol{\omega}^2 I_G . \quad (28)$$

### Example

### Cylinder on a ramp

Let us consider a uniform cylinder of weight  $W$  and radius  $R$  rolling without slipping down a ramp of angle  $\phi$ .



We consider the fixed reference frame,  $x, y$ , instantaneously located at the center of mass as shown in the figure. The equations of motion, 22 and 23, are, in this case,

$$\begin{aligned} m\ddot{x}_G &= W \sin \phi - F \\ m\ddot{y}_G &= N - W \cos \phi \\ I_G \alpha &= -FR \end{aligned}$$

In these equations,  $W = mg$ , but the normal force,  $N$ , and the friction force,  $F$ , are unknown. These two additional unknowns can be determined if we provide two additional kinematic conditions. First, we have that  $\ddot{y}_G = 0$ , from which we can determine  $N$  as

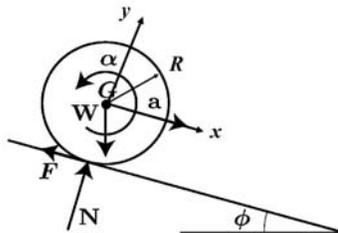
$$N = W \cos \phi .$$

Second, since the cylinder rolls without sliding, we have that  $\ddot{x}_G = -R\alpha$ . Solving for  $\ddot{x}_G$ , we obtain

$$\ddot{x}_G = \frac{g \sin \phi}{1 + (I_G/mR^2)} , \quad (29)$$

and  $F = (I_G \ddot{x}_G)/R^2$ . For the uniform cylinder, we have that  $I_G = mR^2/2$  and  $\ddot{x}_G = (2g \sin \phi)/3$ . If instead of having a uniform disc, we had a uniform ring with all the mass concentrated at the rim, then  $I_G = mR^2$  and  $\ddot{x}_G = (g \sin \phi)/2$ . Also, if there was no friction,  $F$  would be zero, the cylinder would not rotate, and the acceleration would be that of a sliding mass point  $\ddot{x}_G = g \sin \phi$ .

We now consider that the total cylinder mass is located as a mass point in the center of the cylinder. Then the moment of inertia about  $G$  is zero. The cylinder will roll down the ramp but it requires no moment to cause this motion.



Writing the equations about the center of mass we obtain the result that since the moment of inertia is zero, the moment about  $G$  must be zero. Since both  $g$  and  $N$  are directed towards the center of mass, the only moment comes from the friction  $F$ ; therefore,  $F$  must be zero.

## Rotation about a Fixed Axis

For cases in which there is a fixed point in the body, the motion of the body can be described with a single parameter (e.g. the rotation angle). In principle, we could still consider equations (22) and (23) and use the kinematic conditions to enforce that the motion of the fixed point is zero. Alternatively, the analysis is often simplified if we consider the conservation of angular momentum about the fixed point directly. In this case, we have,

$$I_O \alpha = M_O . \tag{30}$$

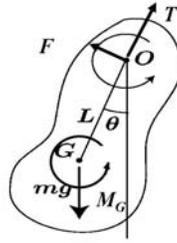
where  $I_O$  is the moment of inertia about **the fixed point**,  $O$ ,  $\alpha$  is the angular acceleration, and  $M_O$  is the sum of external moments about  $O$ .

### Example

### Compound Pendulum

An example of a rigid body rotating about a fixed axis is the compound pendulum. A compound pendulum is a rigid body hinged, without friction, about a horizontal axis offset from its center of mass, and acted upon by its own weight as an external force. The angular velocity of the pendulum about its pivot point is  $\omega = \dot{\theta}$ ; the angular acceleration of the pendulum about its pivot point is  $\alpha = \ddot{\theta}$ . As noted before, every mass

point in the 2D solid body of the pendulum has the identical angular velocity and angular acceleration,  $\omega$  and  $\alpha$ .



It is convenient to apply the conservation on angular momentum about the point  $O$  because the unknown reaction forces at the support contribute no moment about  $O$ . The conservation of angular momentum about  $O$  results in

$$I_O \ddot{\theta} = M_O = -mgL \sin \theta ,$$

or,

$$\ddot{\theta} = -\frac{g}{I_O/mL} \sin \theta .$$

Comparing it with the equation for a simple pendulum from Lecture 10, we see that the motion of a compound pendulum is identical to the motion of a simple pendulum of *equivalent* length,  $L_{equiv}$ ,

$$L_{equiv} = \frac{I_O}{mL} .$$

We now apply conservation of angular momentum about the center of mass to determine the reaction force  $\mathbf{F}$ . We see that gravity produces no moment about  $G$  and the only moments are produced by reaction forces at the attachment point. The tension  $\mathbf{T}$  at the attachment point produces no moment about the center of mass. The force  $\mathbf{F}$  produces a moment  $M_G = FL$  which must equal  $I_G \ddot{\theta}$ . We remark on our earlier observation from Lecture 10 that if the pendulum is a mass point, the moment of inertia about the center of mass is zero. Therefore, there can be no moment exerted by the attachment point about the center of mass. Therefore can be no reaction in the direction normal to the direction from the pivot to the center of mass—which would produce a moment. Therefore the force is aligned with the direction of the tension  $\mathbf{T}$  as in a string. We now apply Newton's law about the center of mass in the direction of  $T$ , balancing the tension and a component of the gravitational force with the centripetal acceleration to determine  $T$ .

$$-T + mg \cos \theta = -mL\dot{\theta}^2 \tag{31}$$

### ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition  
 4/1, 4/2, 4/3 (kinetic energy expression only), 4/4, 4/5 (momentum only) 6/1, 6/2, 6/3, 6/4, 6/5 (angular momentum)

MIT OpenCourseWare  
<http://ocw.mit.edu>

16.07 Dynamics  
Fall 2009

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.