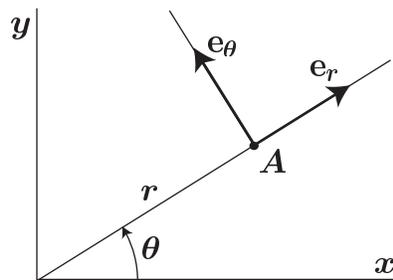


Lecture L5 - Other Coordinate Systems

In this lecture, we will look at some other common systems of coordinates. We will present polar coordinates in two dimensions and cylindrical and spherical coordinates in three dimensions. We shall see that these systems are particularly useful for certain classes of problems.

Polar Coordinates ($r - \theta$)

In polar coordinates, the position of a particle A , is determined by the value of the radial distance to the origin, r , and the angle that the radial line makes with an arbitrary fixed line, such as the x axis. Thus, the trajectory of a particle will be determined if we know r and θ as a function of t , i.e. $r(t), \theta(t)$. The directions of increasing r and θ are defined by the orthogonal unit vectors e_r and e_θ .

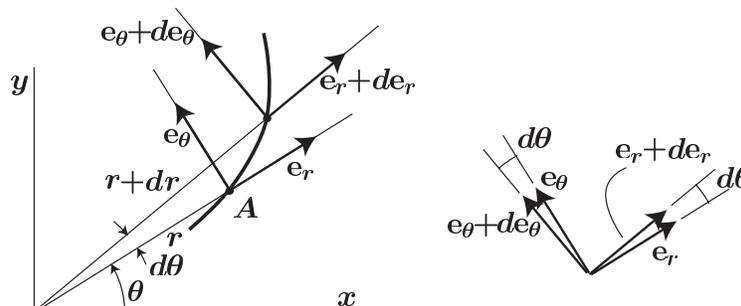


The position vector of a particle has a magnitude equal to the radial distance, and a direction determined by e_r . Thus,

$$\mathbf{r} = r\mathbf{e}_r . \quad (1)$$

Since the vectors e_r and e_θ are clearly different from point to point, their variation will have to be considered when calculating the velocity and acceleration.

Over an infinitesimal interval of time dt , the coordinates of point A will change from (r, θ) , to $(r + dr, \theta + d\theta)$ as shown in the diagram.



We note that the vectors \mathbf{e}_r and \mathbf{e}_θ do not change when the coordinate r changes. Thus, $d\mathbf{e}_r/dr = \mathbf{0}$ and $d\mathbf{e}_\theta/dr = \mathbf{0}$. On the other hand, when θ changes to $\theta + d\theta$, the vectors \mathbf{e}_r and \mathbf{e}_θ are rotated by an angle $d\theta$. From the diagram, we see that $d\mathbf{e}_r = d\theta\mathbf{e}_\theta$, and that $d\mathbf{e}_\theta = -d\theta\mathbf{e}_r$. This is because their magnitudes in the limit are equal to the unit vector as radius times $d\theta$ in radians. Dividing through by $d\theta$, we have,

$$\frac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta, \quad \text{and} \quad \frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_r .$$

Multiplying these expressions by $d\theta/dt \equiv \dot{\theta}$, we obtain,

$$\frac{d\mathbf{e}_r}{dt} \frac{d\theta}{dt} \equiv \frac{d\mathbf{e}_r}{dt} = \dot{\theta}\mathbf{e}_\theta, \quad \text{and} \quad \frac{d\mathbf{e}_\theta}{dt} = -\dot{\theta}\mathbf{e}_r . \quad (2)$$

Note

Alternative calculation of the unit vector derivatives

An alternative, more mathematical, approach to obtaining the derivatives of the unit vectors is to express \mathbf{e}_r and \mathbf{e}_θ in terms of their cartesian components along \mathbf{i} and \mathbf{j} . We have that

$$\begin{aligned} \mathbf{e}_r &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \\ \mathbf{e}_\theta &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} . \end{aligned}$$

Therefore, when we differentiate we obtain,

$$\begin{aligned} \frac{d\mathbf{e}_r}{dr} &= 0, & \frac{d\mathbf{e}_r}{d\theta} &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \equiv \mathbf{e}_\theta \\ \frac{d\mathbf{e}_\theta}{dr} &= 0, & \frac{d\mathbf{e}_\theta}{d\theta} &= -\cos\theta\mathbf{i} - \sin\theta\mathbf{j} \equiv -\mathbf{e}_r . \end{aligned}$$

Velocity vector

We can now derive expression (1) with respect to time and write

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r ,$$

or, using expression (2), we have

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta . \quad (3)$$

Here, $v_r = \dot{r}$ is the *radial velocity* component, and $v_\theta = r\dot{\theta}$ is the *circumferential velocity* component. We also have that $v = \sqrt{v_r^2 + v_\theta^2}$. The radial component is the rate at which \mathbf{r} changes magnitude, or stretches, and the circumferential component, is the rate at which \mathbf{r} changes direction, or swings.

Acceleration vector

Differentiating again with respect to time, we obtain the acceleration

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{r} \mathbf{e}_r + \dot{r} \dot{\mathbf{e}}_r + \dot{r} \dot{\theta} \mathbf{e}_\theta + r \ddot{\theta} \mathbf{e}_\theta + r \dot{\theta} \dot{\mathbf{e}}_\theta$$

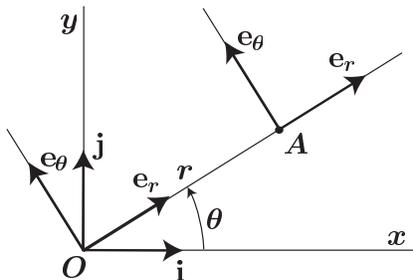
Using the expressions (2), we obtain,

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2) \mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \mathbf{e}_\theta, \quad (4)$$

where $a_r = (\ddot{r} - r\dot{\theta}^2)$ is the *radial acceleration* component, and $a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})$ is the *circumferential acceleration* component. Also, we have that $a = \sqrt{a_r^2 + a_\theta^2}$.

Change of basis

In many practical situations, it will be necessary to transform the vectors expressed in polar coordinates to cartesian coordinates and vice versa.



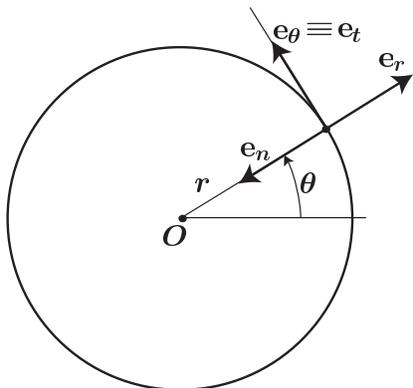
Since we are dealing with free vectors, we can translate the polar reference frame for a given point (r, θ) , to the origin, and apply a standard change of basis procedure. This will give, for a generic vector \mathbf{A} ,

$$\begin{pmatrix} A_r \\ A_\theta \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \end{pmatrix}.$$

Example

Circular motion

Consider as an illustration, the motion of a particle in a circular trajectory having angular velocity $\omega = \dot{\theta}$, and angular acceleration $\alpha = \dot{\omega}$.



In polar coordinates, the equation of the trajectory is

$$r = R = \text{constant}, \quad \theta = \omega t + \frac{1}{2}\alpha t^2.$$

The velocity components are

$$v_r = \dot{r} = 0, \quad \text{and} \quad v_\theta = r\dot{\theta} = R(\omega + \alpha t) = v,$$

and the acceleration components are,

$$a_r = \ddot{r} - r\dot{\theta}^2 = -R(\omega + \alpha t)^2 = -\frac{v^2}{R}, \quad \text{and} \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = R\alpha = a_t,$$

where we clearly see that, $a_r \equiv -a_n$, and that $a_\theta \equiv a_t$.

In cartesian coordinates, we have for the trajectory,

$$x = R \cos(\omega t + \frac{1}{2}\alpha t^2), \quad y = R \sin(\omega t + \frac{1}{2}\alpha t^2).$$

For the velocity,

$$v_x = -R(\omega + \alpha t) \sin(\omega t + \frac{1}{2}\alpha t^2), \quad v_y = R(\omega + \alpha t) \cos(\omega t + \frac{1}{2}\alpha t^2),$$

and, for the acceleration,

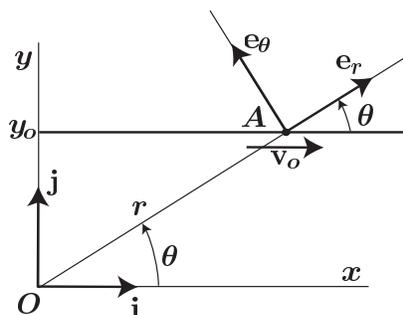
$$a_x = -R(\omega + \alpha t)^2 \cos(\omega t + \frac{1}{2}\alpha t^2) - R\alpha \sin(\omega t + \frac{1}{2}\alpha t^2), \quad a_y = -R(\omega + \alpha t)^2 \sin(\omega t + \frac{1}{2}\alpha t^2) + R\alpha \cos(\omega t + \frac{1}{2}\alpha t^2).$$

We observe that, for this problem, the result is much simpler when expressed in polar (or intrinsic) coordinates.

Example

Motion on a straight line

Here we consider the problem of a particle moving with constant velocity v_0 , along a horizontal line $y = y_0$.



Assuming that at $t = 0$ the particle is at $x = 0$, the trajectory and velocity components in cartesian coordinates are simply,

$$\begin{aligned} x &= v_0 t & y &= y_0 \\ v_x &= v_0 & v_y &= 0 \\ a_x &= 0 & a_y &= 0. \end{aligned}$$

In polar coordinates, we have,

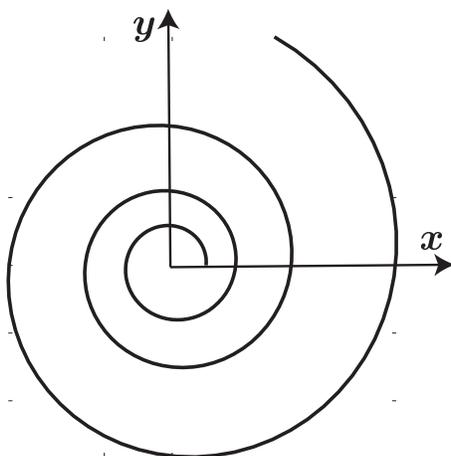
$$\begin{aligned} r &= \sqrt{v_0^2 t^2 + y_0^2} & \theta &= \tan^{-1}\left(\frac{y_0}{v_0 t}\right) \\ v_r &= \dot{r} = v_0 \cos \theta & v_\theta &= r\dot{\theta} = -v_0 \sin \theta \\ a_r &= \ddot{r} - r\dot{\theta}^2 = 0 & a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \end{aligned}$$

Here, we see that the expressions obtained in cartesian coordinates are simpler than those obtained using polar coordinates. It is also reassuring that the acceleration in both the r and θ direction, calculated from the general two-term expression in polar coordinates, works out to be zero as it must for constant velocity-straight line motion.

Example

Spiral motion (Kelpner/Kolenkow)

A particle moves with $\dot{\theta} = \omega = \text{constant}$ and $r = r_0 e^{\beta t}$, where r_0 and β are constants.



We shall show that for certain values of β , the particle moves with $a_r = 0$.

$$\begin{aligned} \mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \\ &= (\beta^2 r_0 e^{\beta t} - r_0 e^{\beta t} \omega^2)\mathbf{e}_r + 2\beta r_0 \omega e^{\beta t} \mathbf{e}_\theta \end{aligned}$$

If $\beta = \pm\omega$, the radial part of \mathbf{a} vanishes. It seems quite surprising that when $r = r_0 e^{\beta t}$, the particle moves with zero radial acceleration. The error is in thinking that \ddot{r} makes the only contribution to a_r ; the term $-r\dot{\theta}^2$ is also part of the radial acceleration, and cannot be neglected.

The paradox is that even though $a_r = 0$, the radial velocity $v_r = \dot{r} = r_0 \beta e^{\beta t}$ is increasing rapidly in time. In polar coordinates

$$v_r \neq \int a_r(t) dt ,$$

because this integral does not take into account the fact that \mathbf{e}_r and \mathbf{e}_θ are functions of time.

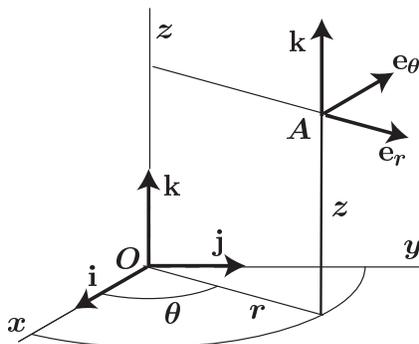
Equations of Motion

In two dimensional polar $r\theta$ coordinates, the force and acceleration vectors are $\mathbf{F} = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta$ and $\mathbf{a} = a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta$. Thus, in component form, we have,

$$\begin{aligned} F_r &= m a_r = m (\ddot{r} - r\dot{\theta}^2) \\ F_\theta &= m a_\theta = m (r\ddot{\theta} + 2\dot{r}\dot{\theta}) . \end{aligned}$$

Cylindrical Coordinates ($r - \theta - z$)

Polar coordinates can be extended to three dimensions in a very straightforward manner. We simply add the z coordinate, which is then treated in a cartesian like manner. Every point in space is determined by the r and θ coordinates of its projection in the xy plane, and its z coordinate.



The unit vectors \mathbf{e}_r , \mathbf{e}_θ and \mathbf{k} , expressed in cartesian coordinates, are,

$$\begin{aligned} \mathbf{e}_r &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \\ \mathbf{e}_\theta &= -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \end{aligned}$$

and their derivatives,

$$\dot{\mathbf{e}}_r = \dot{\theta} \mathbf{e}_\theta, \quad \dot{\mathbf{e}}_\theta = -\dot{\theta} \mathbf{e}_r, \quad \dot{\mathbf{k}} = \mathbf{0} .$$

The kinematic vectors can now be expressed relative to the unit vectors \mathbf{e}_r , \mathbf{e}_θ and \mathbf{k} . Thus, the position vector is

$$\mathbf{r} = r \mathbf{e}_r + z \mathbf{k} ,$$

and the velocity,

$$\mathbf{v} = \dot{r} \mathbf{e}_r + r\dot{\theta} \mathbf{e}_\theta + \dot{z} \mathbf{k} ,$$

where $v_r = \dot{r}$, $v_\theta = r\dot{\theta}$, $v_z = \dot{z}$, and $v = \sqrt{v_r^2 + v_\theta^2 + v_z^2}$. Finally, the acceleration becomes

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2) \mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \mathbf{e}_\theta + \ddot{z} \mathbf{k} ,$$

where $a_r = \ddot{r} - r\dot{\theta}^2$, $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$, $a_z = \ddot{z}$, and $a = \sqrt{a_r^2 + a_\theta^2 + a_z^2}$.

Note that when using cylindrical coordinates, r is not the modulus of \mathbf{r} . This is somewhat confusing, but it is consistent with the notation used by most books. Whenever we use cylindrical coordinates, we will write $|\mathbf{r}|$ explicitly, to indicate the modulus of \mathbf{r} , i.e. $|\mathbf{r}| = \sqrt{r^2 + z^2}$.

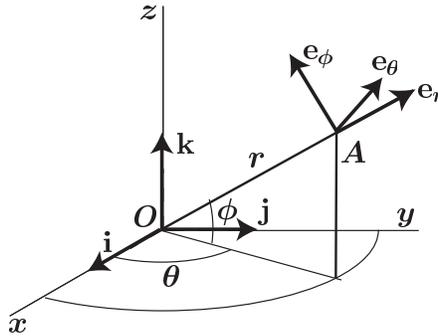
Equations of Motion

In cylindrical $r\theta z$ coordinates, the force and acceleration vectors are $\mathbf{F} = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_z \mathbf{e}_z$ and $\mathbf{a} = a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta + a_z \mathbf{e}_z$. Thus, in component form we have,

$$\begin{aligned} F_r &= m a_r = m (\ddot{r} - r\dot{\theta}^2) \\ F_\theta &= m a_\theta = m (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \\ F_z &= m a_z = m \ddot{z} . \end{aligned}$$

Spherical Coordinates ($r - \theta - \phi$)

In spherical coordinates, we utilize two angles and a distance to specify the position of a particle, as in the case of radar measurements, for example.



The unit vectors written in cartesian coordinates are,

$$\begin{aligned} \mathbf{e}_r &= \cos \theta \cos \phi \mathbf{i} + \sin \theta \cos \phi \mathbf{j} + \sin \phi \mathbf{k} \\ \mathbf{e}_\theta &= -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \\ \mathbf{e}_\phi &= -\cos \theta \sin \phi \mathbf{i} - \sin \theta \sin \phi \mathbf{j} + \cos \phi \mathbf{k} \end{aligned}$$

The derivation of expressions for the velocity and acceleration follow easily once the derivatives of the unit vectors are known. In three dimensions, the geometry is somewhat more involved, but the ideas are the same. Here, we give the results for the derivatives of the unit vectors,

$$\dot{\mathbf{e}}_r = \dot{\theta} \cos \phi \mathbf{e}_\theta + \dot{\phi} \mathbf{e}_\phi , \quad \dot{\mathbf{e}}_\theta = -\dot{\theta} \cos \phi \mathbf{e}_r + \dot{\theta} \sin \phi \mathbf{e}_\phi , \quad \dot{\mathbf{e}}_\phi = -\dot{\phi} \mathbf{e}_r - \dot{\theta} \sin \phi \mathbf{e}_\theta ,$$

and for the kinematic vectors

$$\begin{aligned}
 \mathbf{r} &= r \mathbf{e}_r \\
 \mathbf{v} &= \dot{r} \mathbf{e}_r + r \dot{\theta} \cos \phi \mathbf{e}_\theta + r \dot{\phi} \mathbf{e}_\phi \\
 \mathbf{a} &= (\ddot{r} - r \dot{\theta}^2 \cos^2 \phi - r \dot{\phi}^2) \mathbf{e}_r \\
 &\quad + (2\dot{r} \dot{\theta} \cos \phi + r \ddot{\theta} \cos \phi - 2r \dot{\theta} \dot{\phi} \sin \phi) \mathbf{e}_\theta \\
 &\quad + (2\dot{r} \dot{\phi} + r \dot{\phi}^2 \sin \phi \cos \phi + r \ddot{\phi}) \mathbf{e}_\phi .
 \end{aligned}$$

Equations of Motion

Finally, in spherical $r\theta\phi$ coordinates, we write $\mathbf{F} = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi$ and $\mathbf{a} = a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta + a_\phi \mathbf{e}_\phi$. Thus,

$$\begin{aligned}
 F_r &= m a_r = m (\ddot{r} - r \dot{\theta}^2 \cos^2 \phi - r \dot{\phi}^2) \\
 F_\theta &= m a_\theta = m (2\dot{r} \dot{\theta} \cos \phi + r \ddot{\theta} \cos \phi - 2r \dot{\theta} \dot{\phi} \sin \phi) \\
 F_\phi &= m a_\phi = m (2\dot{r} \dot{\phi} + r \dot{\phi}^2 \sin \phi \cos \phi + r \ddot{\phi}) .
 \end{aligned}$$

Application Examples

We will look at some applications of Newton's second law, expressed in the different coordinate systems that have been introduced. Recall that Newton's second law

$$\mathbf{F} = m \mathbf{a} , \tag{5}$$

is a vector equation which is valid for inertial observers.

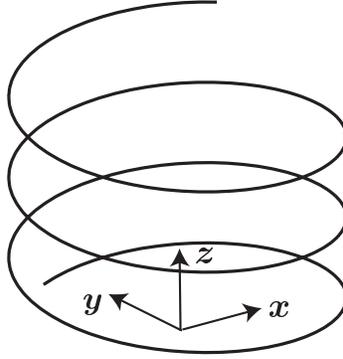
In general, we will be interested in determining the motion of a particle given that we know the external forces. Equation (5), written in terms of either velocity or position, is a differential equation. In order to calculate the velocity and position as a function of time we will need to integrate this equation either analytically or numerically. On the other hand, the reverse problem of computing the forces given motion is much easier and only requires direct evaluation of (5). It is also common to have mixed type problems, in which we know some components of the force and some components of the acceleration. The goal is then to determine the remaining unknown terms.

While no general rules can be given regarding the appropriate choice of a coordinate system, we note that intrinsic coordinates are particularly useful in constrained problems, where the trajectory is known beforehand.

Example

Aircraft flying on a helix

A 10,000 lb aircraft is descending on a cylindrical helix. The rate of descent is $\dot{z} = -10\text{ft/s}$, the speed is $v = 211 \text{ ft/s}$, and $\dot{\theta} = 3^\circ \approx 0.05\text{rad/s}$. This is standard for gas turbine powered aircraft. We want to know the force on the aircraft and the radius of curvature of the path.



We have,

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{e}_z = v\mathbf{e}_t$$

Since, $r = R$, $\dot{r} = 0$. Therefore, $211 = \sqrt{(0.05R)^2 + 10^2}$, or $R = 4,215$ ft. For the acceleration,

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta + \ddot{z}\mathbf{e}_z = \dot{v}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n,$$

and, considering only the non-zero terms,

$$\mathbf{a} = -R\dot{\theta}^2\mathbf{e}_r = \frac{v^2}{\rho}\mathbf{e}_n.$$

We see that $\mathbf{e}_n = -\mathbf{e}_r$, and that,

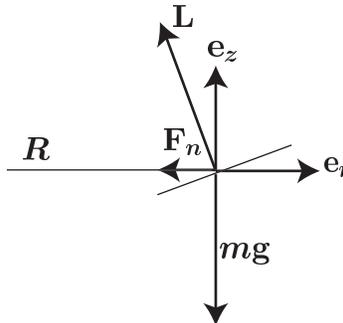
$$a = (0.05)^2 4,215 = 10.54 \text{ ft/s}^2 = \frac{v^2}{\rho}, \quad \rho = \frac{211}{10.54} = 4,225 \text{ ft}.$$

The normal force on the aircraft is

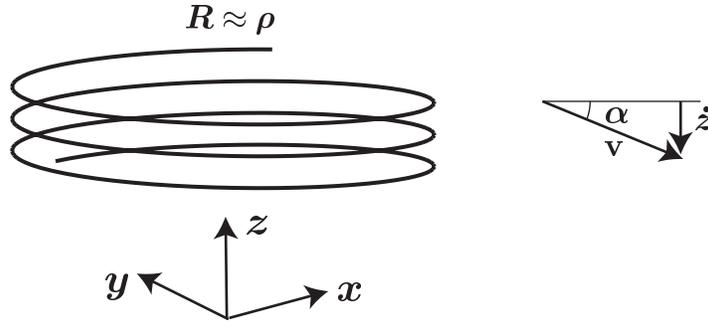
$$F_n = ma_n = \frac{10,000}{32} 10.54 = 3,273 \text{ lb},$$

and finally, the lift, \mathbf{L} , is

$$\mathbf{L} = -3,273 \mathbf{e}_r + 10,000 \mathbf{e}_z \text{ lb}.$$



Here we see that $\rho \approx r$ which means that the helix is very tight.

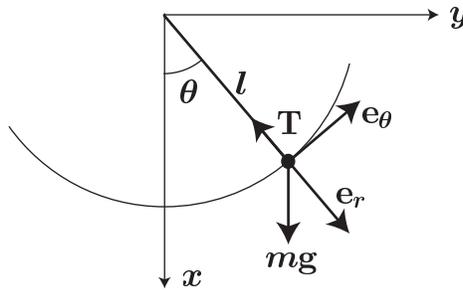


The angle of descent α is calculated as $\sin \alpha = -\dot{z}/v$, or, $\alpha = -2.72^\circ$. This angle is sometimes called the *pitch* of the helix.

Example

Pendulum

Now, we consider a simple pendulum consisting of a mass, m , suspended from a string of length l and negligible mass.



We can formulate the problem in polar coordinates, and noting that $r = l$ (constant), write for the r and θ components,

$$\begin{aligned} mg \cos \theta - T &= -ml\dot{\theta}^2 \\ -mg \sin \theta &= ml\ddot{\theta} , \end{aligned} \tag{6}$$

where T is the tension on the string. If we restrict the motion to small oscillations, we can approximate $\sin \theta \approx \theta$, and the θ -equation becomes

$$\ddot{\theta} + \frac{g}{l}\theta = 0 .$$

Integrating we obtain the general solution,

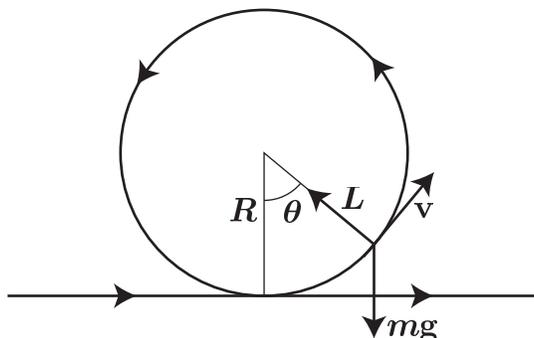
$$\theta(t) = C_1 \cos(\sqrt{\frac{g}{l}} t) + C_2 \sin(\sqrt{\frac{g}{l}} t) ,$$

where the constants C_1 and C_2 are determined by the initial conditions. Thus, if $\theta(0) = \theta_{\max}$,

$$\theta(t) = \theta_{\max} \cos(\sqrt{\frac{g}{l}} t) .$$

Example**Aircraft flying a perfect loop** (Hollister)

Consider an aircraft flying a perfect loop, i.e. a circle in the vertical plane. Assume that the engine thrust exactly cancels the aerodynamic drag so that the lift and gravity are the only unbalanced forces on the aircraft. This assumption makes the problem into the same dynamical model that we have used in the previous example.



Since the lift, \mathbf{L} , is perpendicular to the flight path, we have that the force on the aircraft, in normal and tangential components, is

$$\mathbf{F} = -mg \sin \theta \mathbf{e}_t + (L - mg \cos \theta) \mathbf{e}_n .$$

Thus,

$$\begin{aligned} a_t &= \dot{v} = r\ddot{\theta} = -g \sin \theta \\ a_n &= \frac{v^2}{R} = \frac{L}{m} - g \cos \theta . \end{aligned} \quad (7)$$

Since, $v dv = a_t ds = a_t R d\theta = -Rg \sin \theta d\theta$. Thus, integrating,

$$v^2 = v_0^2 + 2Rg(\cos \theta - 1) , \quad (8)$$

where v_0 is the velocity at the bottom of the loop when $\theta = 0$. To be able to go over the top we need $v > 0$ when $\theta = \pi$. This means that we need $v_0 > 2\sqrt{Rg}$.

Note that for $v_0 < 2\sqrt{Rg}$, we can calculate the maximum angle the aircraft can reach, θ_{max} . If we set $v = 0$ when $\theta = \theta_{max}$, we have,

$$\theta_{max} = \cos^{-1}\left(1 - \frac{v_0^2}{2Rg}\right) .$$

The necessary lift, L , can be calculated as a function of θ . From (7) and (8), we have

$$\frac{L}{m} = \frac{v^2}{R} + g \cos \theta = \frac{v_0^2}{R} + 3g \cos \theta - 2g .$$

We have that, in order for θ to go from 0 to π , the aircraft has to have a range of lift capability that extends over $5g$.

It turns out that most aircraft do not have this capability and consequently do not fly perfect loops.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

2/6, 2/7, 3/5

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