16.06 Principles of Automatic Control
Lecture 8

The Routh Stability Criterion

Suppose we have a transfer function

$$T(s) = \frac{Y(s)}{R(s)} \frac{b_0 s^m + b_1 S^{m-1} + ... + b_m}{s^n + a_1 s^{n-1} + ... + a_n}$$

We can always factor as

$$T(s) = \kappa \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}$$

The closed-loop system is stable if

$$\Re(p_i) < 0, \forall\ i$$

NB: It might turn out that there are pole-zero cancellations, that is, $z_i = p_j$ for some $i.j$.
If this happens, system is still unstable if $\Re(p_j) > 0$.
The *characteristic equation* is:

$$\phi(s) = s^n + a_1 s^{n-1} + ... + a_{n-1}s + a_n = 0$$

The roots are, of course, $p_1, p_2, ...p_n$.

Important question:
Can we tell if the system is stable, *without* actually solving for the roots?

Partial answer: A necessary condition for all the roots to be stable is that all the coefficients of $\phi(s)$ be positive. So if at least one coefficient is negative, system must be unstable. A complete answer to the question is obtained using the *Routh Array*. The array is constracted as bellow:

$$\begin{array}{lcccc} \text{Row n:} & 1 & a_2 & a_4 & \cdots \\ \text{Row n-1:} & a_1 & a_3 & a_5 & \cdots \\ \text{Row n-2:} & b_1 & b_2 & b_3 & \cdots \\ \text{Row n-3:} & c_1 & c_2 & c_3 & \cdots \\ \quad\vdots & & \ddots & & \\ \text{Row 2:} & * & * & & \\ \text{Row 1:} & * & & & \\ \text{Row 0:} & * & & & \end{array}$$

The first two rows come directly from the polynomial $\phi(x)$. Each subsequent row is formed by operations on the two rows above:

$$b_1 = -\frac{\begin{vmatrix} 1 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1} = \frac{a_1 a_2 - a_3}{a_1}$$

$$b_2 = -\frac{\begin{vmatrix} 1 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1} = \frac{a_1 a_4 - a_5}{a_1}$$

$$c_1 = -\frac{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1} = \frac{b_1 a_3 - a_1 b_2}{b1}$$

The number of unstable poles is the number of sign changes in the first column of the array.

Example:

$$\phi(s) = s^3 + 2s^2 + 3s + 8$$

The Routh Array is
H_0 is

$$\begin{array}{rrrr} 3: & 1 & 3 & 0 \\ 2: & 2 & 8 & 0 \\ 1: & -1 & 0 & \\ 0: & 8 & & \\ & \downarrow & & \end{array}$$

First column has two sign changes!

There are two unstable poles. In fact, the roots are:

-2.2483

$0.1241 + 1.8822i$

$0.1241 - 1.8822i$

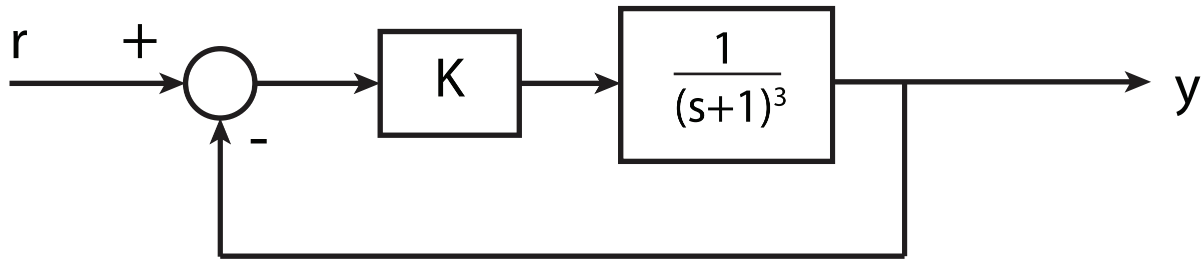
Note: We can scale any row of the array by a positive constant, and not change the sign of any of the terms. This can simplify the algebra by eliminating fractions.

Stability vs. Parameter Range

It's much easier to use a calculator or Matlab to find roots. So why use Routh?
Routh allows us to determine symbolically what values of a parameter will lead to stability/instability.

Example:

For what values of k is the following system stable?



Solution:
The Closed Loop transfer function is:

$$
\begin{aligned}
T(s) =& \frac{KG(s)}{1+KG(s)} = \frac{\frac{K}{(s+1)^3}}{1+\frac{K}{(s+1)^3}} \\
=& \frac{K}{(s+1)^3+K} \\
\Rightarrow \phi(s) =& s^3+3s^2+3s+1+K
\end{aligned}
$$

The Routh array is

$$
\begin{array}{cccc}
3: & 1 & 3 & 0 \\
2: & 3 & 1+K & 0 \\
1: & \frac{8-K}{3} & 0 & \\
0: & 1+K & 0 &
\end{array}
$$

For stability, need first column to be positive, so that $K < 8$ and $K > -1$.
If $K < -1$, first column is $+++-$, so there is *1* unstable pole.
If $K > 8$, first column is $++-+$, so there are 2 unstable poles.

Possible problems:

If the first element of a row is zero, process fails.

Solution: Replace 0 by ϵ, a small positive number.
If a whole row is zero, must replace row as explained in the book. This happens whenever there is a complex conjugate pair of roots on the imaginary axis.