16.06 Principles of Automatic Control
Lecture 32

(continued from previous lecture..)

Note the much improved response. Now we have:

$$M_p = 0.246 \quad \text{(cont.)}$$
$$M_p = 0.231 \quad \text{(disc.)}$$

- very close!

Full cycle processing delay

What happens when processing delay is a full period, T? In the textbook, this case is described as requiring that the numerator of K_d must have one less power of z than the denominator. However, in the emulation approach, this might be tricky.

The solution is to account for the delay in G_d by adding the factor $1/z$, which is the z-transform of a one sample delay. So for the previous example, we would have:

$$G_d(z) = 0.0003099 \, \frac{z + 0.9917}{(z-1)(z-0.9753)z}$$

So now we have an effective delay of $\frac{3}{2}T$, $\frac{1}{2}T$ from the ZOH, and T from the processing delay. Let's redo the design assuming this larger delay:

$$\begin{aligned}\angle K(j10) =& -G(j10) - 180^\circ + \text{PM} + \frac{3}{2}\omega_c T \cdot \frac{180^\circ}{\pi} \\ =& 65.9^\circ\end{aligned}$$

So the lead ratio satisfies

$$\sqrt{\frac{b}{a}} = 4.685$$

So let

$$b = 47 \text{ r/s}$$
$$a = 2.1 \text{ r/s}$$

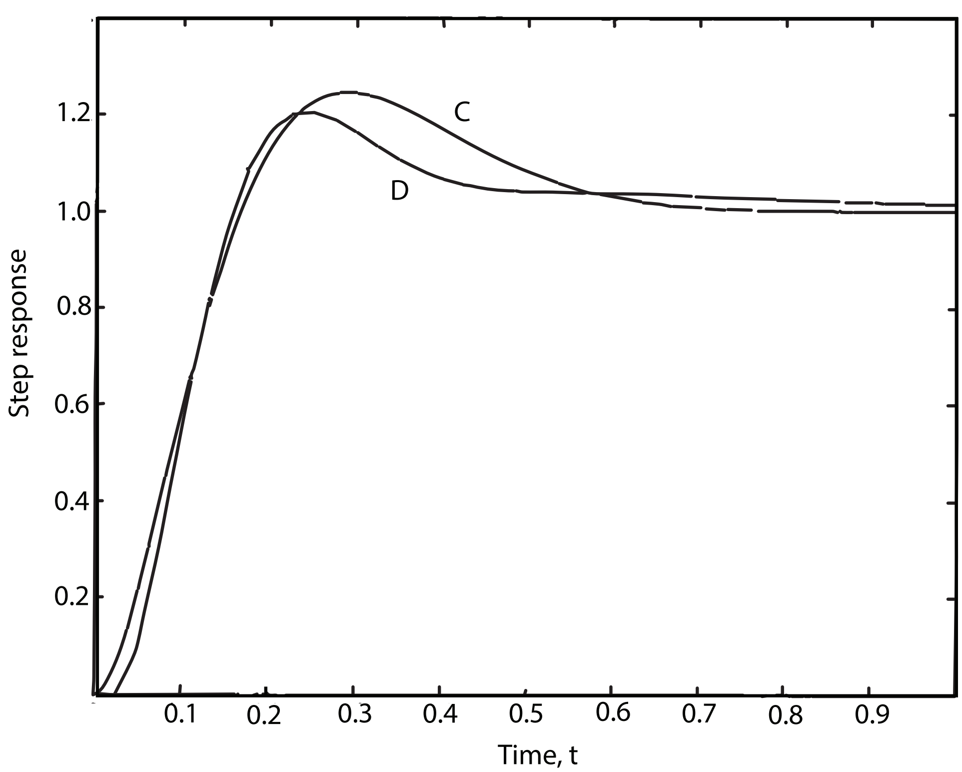
The continuous compensator is

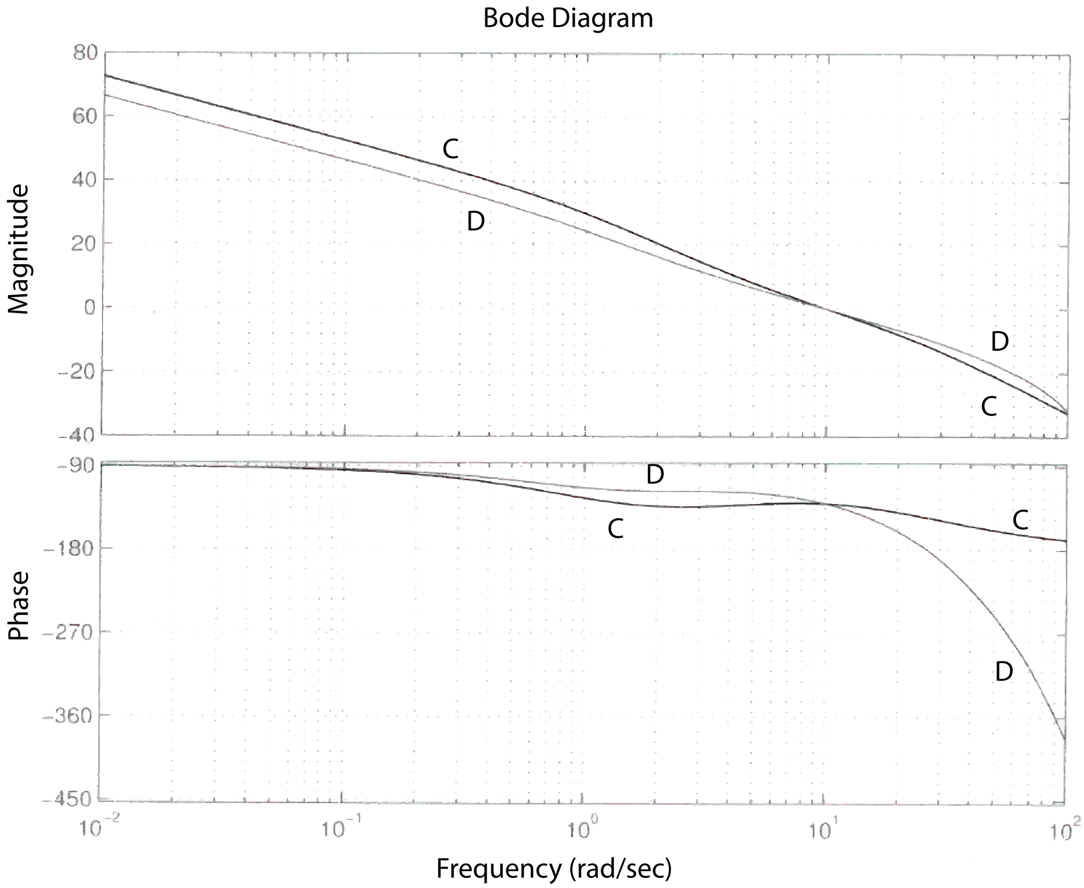
$$K(s) = 21.1 \, \frac{1 + s/2.1}{1 + s/47}$$

Using the Tustin transform, we obtain:

$$K_d(z) = 305.3 \, \frac{z - 0.9488}{z - 0.2598}$$

See the step response below. Why is the continuous and discrete response so different? $K(s)$ for the continuous system and $K_d(s)$ for the discrete system are different - they have different DC gains and different lag ratios. As a result, the Bode plots for the loop gains are significantly different (see the plot below).





Discrete Design

Can also design directly in z-plane, using root locus or other tools.

When we designed $K_d(z)$ using emulation methods, we needed $G_d(z)$ only to rest our result, not to do the design, for which we only needed $G(s)$. (We also calculated the equivalent delay, $T/2$ or $3T/2$.)

For discrete design, however, we need $G_d(z)$, which may be calculated as

$$G_d(z) = (1 - z^{-1})\mathcal{Z}\{\frac{G(s)}{s}\}$$

Example

$$G(s) = \frac{1}{s(s+1)}, \quad T = 0.025$$

$$
\begin{aligned}
\frac{G(s)}{s} =& \frac{1}{s^2(s+1)} = \frac{1}{s+1} + \frac{1}{s^2} - \frac{1}{s} \\
\mathcal{Z}\ \ \frac{G(s)}{s}\} =& \frac{z}{z-e^{-T}} + \frac{Tz}{(z-1)^2} - \frac{z}{z-1}
\end{aligned}
$$

$$
\begin{aligned}
G_d(z) =& \frac{z-1}{z}\mathcal{Z}\ \ \frac{G(s)}{s}\} \\
=& \frac{z-1}{z-e^{-T}} + \frac{T}{z-1} - 1 \\
=& \frac{z-1}{z-0.9753} + \frac{0.025}{z-1} - 1 \\
=& \frac{0.0003099z + 0.0003073}{z^2 - 1.9753z + 0.9753}
\end{aligned}
$$

as before.

The basic control laws:

Proportional:

$$
u[k] = K_p e[k] \ \Rightarrow \ K(z) = K_p
$$

Derivative:

$$
\begin{aligned}
u[k] =& K_D \frac{(e[k] - e[k-1])}{T} \\
=& k_D(e[k] - e[k-1]) \\
\Rightarrow K_d(z) =& k_D(1 - z^{-1}) = k_d \frac{z-1}{z}
\end{aligned}
$$

Integral:

$$
\begin{aligned}
u[k] =& u[k-1] + K_I \cdot T\ e[k] \\
\Rightarrow (1 - z^{-1})U(z) =& K_I \cdot T\ E(z) \\
\Rightarrow K_d(z) =& K_I T\ \frac{z}{z-1} = k_I \frac{z}{z-1}
\end{aligned}
$$

Lead:

$$
K_d(z) = k\ \frac{z-\alpha}{z-\beta}, \quad \alpha > \beta
$$

Example

Design a digital controller for the plant

$$G(s) = \frac{1}{s^2}$$

with sample period $T = 1$ sec, so that $\omega_n \approx 0.3$ r/s, and $\zeta = 0.7$.

The discretization plant is

$$G_d(z) = \frac{T^2}{2}\frac{z+1}{(z-1)^2} = \frac{1}{2}\frac{z+1}{(z-1)^2}$$

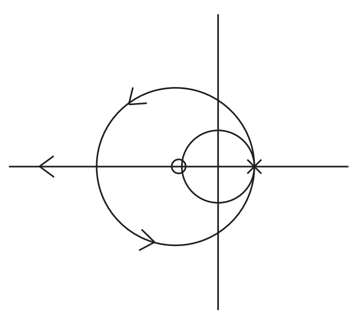
We want the closed loop poles to be at

$$\begin{aligned} z =& e^{sT} \\ s_1 =& -\zeta\omega_n \pm j\sqrt{1-\rho^2}\omega_n \\ =& -0.21 \pm j0.21 \end{aligned}$$

So we want the dominant closed-loop poles at

$$z = 0.791 \pm 0.170j \text{ (book slightly off)}$$

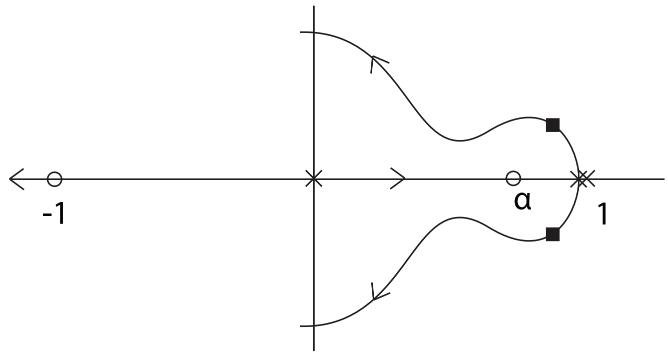
Proportional gain does not work, since the locus is



which is entirely outside the unit circle. Instead, use a PD controller, which will be of the form

$$K_d(z) = k\frac{z-\alpha}{z}$$

So new root locus will be:



Choose α to get angle condition right

$$\begin{aligned}\phi_1 = &\phi_2 = \angle(z-1) = 140.9^\circ \\ \phi_3 = &\angle(z-0) = 12.2^\circ \\ \Psi_1 = &\angle(z-(-1)) = 5.4^\circ\end{aligned}$$

We need

$$\begin{aligned}-\phi_1 - &\phi_2 - \phi_3 + \Psi_1 + \Psi_2 = -180 \\ \Rightarrow \Psi_2 = &108.4^\circ = \angle(z-\alpha) \\ &= \tan^{-1}\left(\frac{0.170}{0.791-\alpha}\right) \\ \Rightarrow \alpha = &+0.8475\end{aligned}$$

So controller is

$$K_d(z) = k\ \frac{z-0.8475}{z}$$

To find k, use gain condition

$$|K_d(z)G_d(z)| = 1 \text{ at C. L. pole}$$

$$\Rightarrow k = 0.3647,$$

$$\boxed{K_d(z) = 0.3647\ \frac{z-0.8475}{z}}$$

The step response is shown below. Notice that the overshoot is significant - much more than would be expected with $\zeta = 0.707$, due to zero of the compensator.

So let's put derivative term in a minor loop: