16.06 Principles of Automatic Control
Lecture 30

Z-transform Inversion

There are 3 ways to invert a Z-transform:

1. Partial Fraction Expansion

Example

$$F(z) = \frac{z}{(z-1/2)(z-1/3)}$$

Using coverup method:

$$F(z) = \frac{3}{z-1/2} - \frac{z}{z-1/3}$$

But we don't know the inverse transform of $\frac{1}{z-a}$.

Instead, do PFE as

$$\begin{aligned}
\frac{F(z)}{z} &= \frac{1}{(z-1/2)(z-1/3)} \\
&= \frac{6}{z-1/2} - \frac{6}{z-1/3} \\
\Rightarrow F(z) &= \frac{6z}{z-1/2} - \frac{6z}{z-1/3} \\
\Rightarrow f[k] &= \left((1/2)^k - (1/3)^k\right)\sigma[k]
\end{aligned}$$

2. Inverse transform integral

$$f[k] = \frac{1}{2\pi j} \oint F(z) z^{k-1} dz$$

where the integral is counter-clockwise around the origin in the region of convergence. If the integral is done using residues, this method reduces to method 1.

3. Long division

There is no analog to this in continuous time![1]

By expanding $F(z)$ in powers of $1/z$, can obtain samples $f[k]$ directly.

Example:

$$F(z) = \frac{z}{z-a} = \frac{1}{1-az}$$

Do long division:

$$\begin{array}{rl} & \underline{1 + az^{-1} 1 + a^2 z^{-2} + \cdots} \\ 1 - az^{-1} & |1 \\ & \underline{1 - az^{-1}} \\ & \quad +az^{-1} \\ & \quad \underline{+az^{-1} - a^2 z^{-2}} \\ & \qquad\qquad +a^2 z^{-2} \\ & \qquad\qquad \underline{+a^2 z^{-2} - a^3 z^{-3}} \\ & \qquad\qquad\qquad\qquad \vdots \end{array}$$

So $f[k] = a^k,\ k \geqslant 0$.

In practice, not very practical[2], but can be easily implemented to directly obtain, say, step response.

Relationship between s and z.

Consider a continuous-time signal

$$f(t) = \sigma(t) e^{-at}$$

[1]Actually, expanding $F(s)$ in powers of $1/s$ yields the infinite series (Taylor series) for $f(t)$, which isn't really that useful.

[2]Not very practical for *hand* computation.

Its Laplace transform is

$$F(s) = \frac{1}{s+a}$$

with pole at $s = -a$. The z-transform of the sampled signal $f(kT)$ is

$$F(z) = \frac{z}{z - e^{-et}}$$

with pole at $z = e^{-at}$. Therefore, there is a natural mapping

$$\boxed{z = e^{st}}$$

between s-plane and the z-plane. For example, if we want a system to have natural frequency ω_n and damping ratio ζ, in the z-plane the poles should be at

$$z = e^{\left(-\zeta\omega_n + j\sqrt{1-\zeta^2}\omega_n\right)T}$$

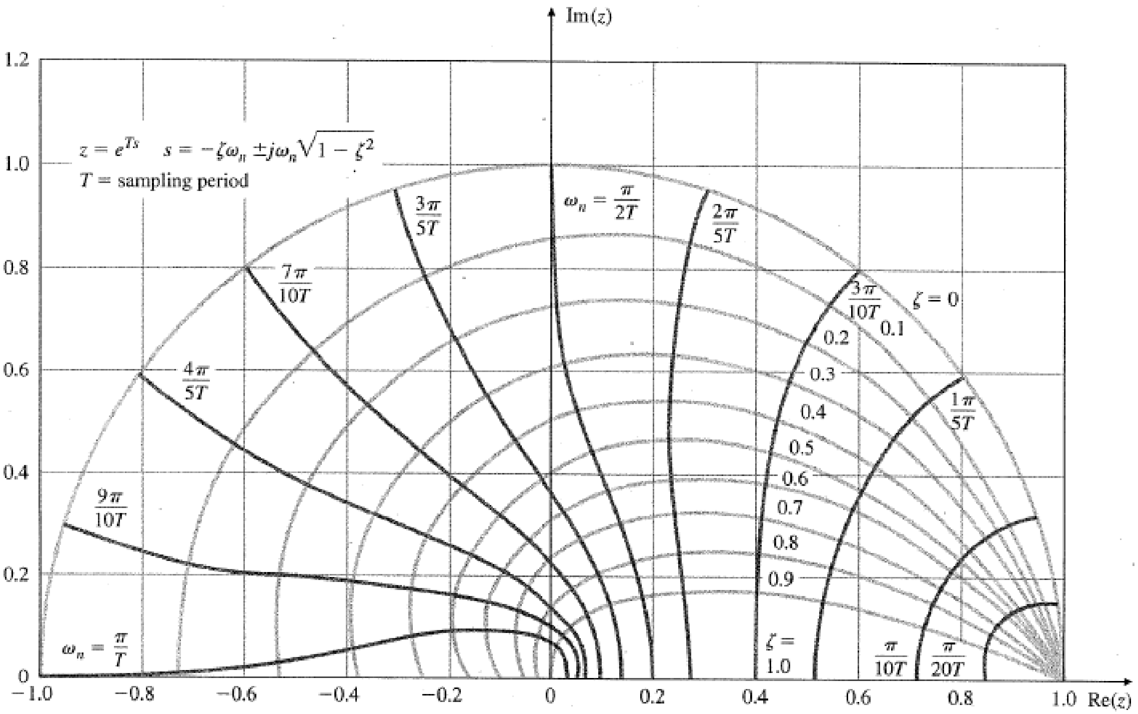


Figure 8.4 Natural frequency (solid color) and damping loci (light color) in the z-plane; the portion below the Re(z)-axis (not shown) is the mirror image of the upper half shown.

Observations:

1. The stability boundary is $|z| = 1$.

2. The region near $s = 0$ maps to the region $z = 1$. For reasonable sampling rates, this is where all the action is.

3. The z-plane pole locations give response information normalized to the sample rate, not to dimensional time as in s-plane. So the meaning of, say, $z = 0.9$ depends on the sampling rate.

4. $z = -1$ corresponds to $\omega = \omega_s/2$, where $\omega_s = 2\pi/T$ =sample rate in radians/sec. $\omega_s/2$ is the Nyquist frequency.

5. Vertical lines in s-plane (constant $\zeta\omega_n$) correspond to circles centered at $z = 0$ in z-plane.

6. Horizontal lines in s-plane (constant ω_d) correspond to radial lines from $z = 0$ in z-plane.

7. Frequency greater than $\omega_s/2$ overlap lower frequencies in the z-plane. This is called aliasing. So should sample *at least* twice as fast as highest frequency component in $G(s)$ and $r(t)$.

Design by Emulation

Can either design directly using, say, root locus in z-plane, or can design in continuous time, and discretize the continuous controller. Must then verify that the design works, of course.

The book suggests three methods:

1. Tustin's approximation
2. Matched Pole-Zero Method (MPZ)
3. Modified MPZ

Will start with...

Tustin's approximation

We have found that

$$z = e^{sT}$$

We can approximate z by

$$z \approx 1 + sT$$

or we can approximate z^{-1} by

$$z^{-1} = e^{-sT} \approx 1 - sT \quad \Rightarrow \quad z \approx \frac{1}{1 - sT}$$

The more symmetric approximation

$$z \approx \frac{1 + sT/2}{1 - sT/2} \tag{1}$$

has some useful properties. Inverting, we have

$$s \approx \frac{2}{T}\left(\frac{z-1}{z+1}\right) \tag{2}$$

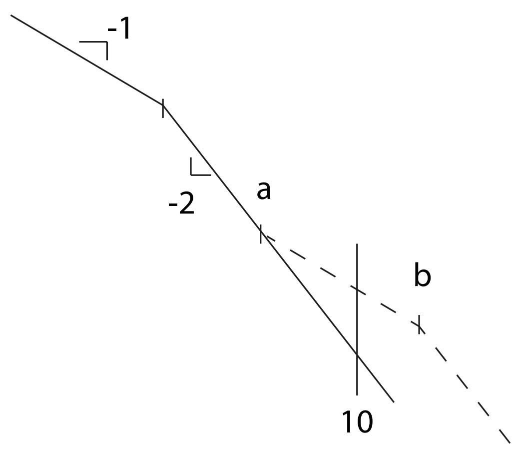
Replacing every occurrence of s by the RHS of (2) in a given $K(s)$ is *Tustin's method*, or the *bilinear transform*. It results in a discrete controller, $K_d(z)$, that is a good approximation to $K(s)$.

Example: For the plant

$$G(s) = \frac{1}{s(s+1)}$$

find a controller for the unity feedback, discrete-time system, with sample time $T = 0.025$ sec (40 Hz sampling), with $\omega_c = 10$ r/s, and PM = 50°.

First, let's design for the continuous system:



Need to add a lead compensator at crossover to get desired PM. Compensator is

$$K(s) = 42.16\,\frac{1 + s/4.2}{1 + s/24}$$

then

$$K_d(z) = K\left(\frac{2}{T}\frac{z-1}{z+1}\right)$$

3 ways to actually compute $K_d(z)$:

1. Actually do the substitution indicated above (ugh!)
2. Use the Matlab command
   $k_d = c2d(k, 0.025,\ 'tustin')$
3. Map the poles and zeros of $K(s)$