16.06 Principles of Automatic Control
Lecture 3

Modeling principles:

1. Identify the states of the system:
   - positions
   - velocities
   - inductor currents
   - capacitor voltages
   - etc
2. Use physics to find dx_1/dt, dx_2/dt,...
3. Organize as:

$$\frac{dx}{dt} = \underline{f}(\underline{x}, u) \qquad y = g(\underline{x}, u)$$

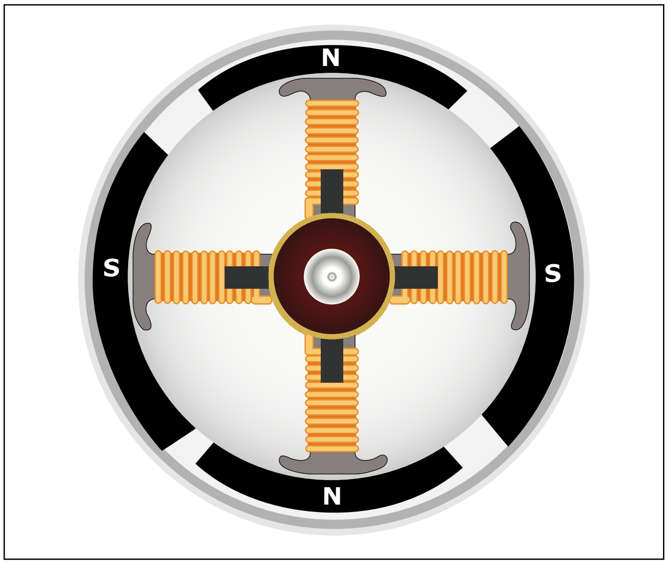
   where

$$\begin{aligned} &x\text{–state vector} \\ &u\text{–control input} \\ &y\text{–output of measurement} \end{aligned}$$

4. Linearize if necessary.

Modeling a DC Motor

Physical layout:



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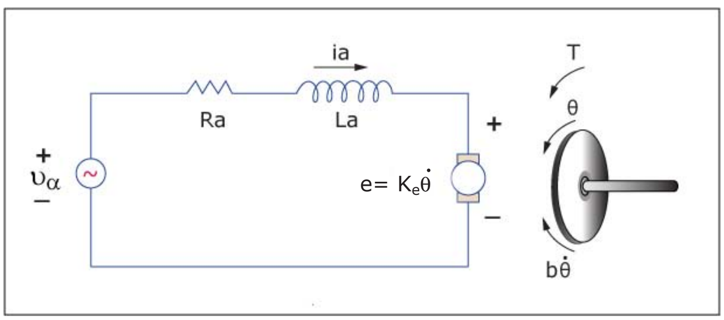
Model:



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The states are:

$$x_1 = \Theta \text{ - motor angle}$$
$$x_2 = \dot{\Theta} \text{ - motor angular velocity}$$
$$x_3 = i_a \text{ - armature current}$$

Find equations of motion:

$$\dot{x}_1 = \frac{d\Theta}{dt} = \dot{\Theta} = x_2 \quad \text{(Kinematics)}$$
$$\dot{x}_2 = \frac{d\dot{\Theta}_1}{dt} = \ddot{\Theta}$$

From free body diagram:

$$
\begin{aligned}
J\ddot{\Theta} =& - b\dot{\Theta} + T \\
-b\dot{\Theta} =& \text{ viscous drag on rotor} \\
T =& \text{ torque due to current} \\
=& K_t i_a, \text{ where } K_t \text{ is a motor torque constant}
\end{aligned}
$$

So

$$
\begin{aligned}
\ddot{\Theta} =& - \frac{b}{J}\dot{\Theta} + \frac{K_t}{J} i_a \\
\dot{x}_2 =& - \frac{b}{J} x_2 + \frac{K_t}{J} x_3
\end{aligned}
$$

Now model the circuit. Start with motor part itself. The power supplied to the motor is

$$P = e i_a$$

This must equal (by 1^{st} law) the torque power:

$$P = T\dot{\Theta} = K_t i_a \dot{\Theta}$$

Equating the previous two equations:

$$e = K_t \dot{\Theta}$$

Therefore,

$$Ke = K_t$$

So now we can find di_a/dt:

$$
\begin{aligned}
\frac{di_a}{dt} =& \frac{1}{L}(v_a - i_a R_a - e) \\
=& \frac{1}{L}(v_a - i_a R_a - K_t \dot{\Theta})
\end{aligned}
$$

Therefore,

$$\dot{x}_3 = -\frac{K_t}{L}x_2 - \frac{R_a}{L}x_3 + \frac{1}{L}v_a$$

In state-space form:

$$\begin{aligned}\underline{\dot{x}} &= \begin{pmatrix}0 & 1 & 0\\ 0 & -b/J & K_t/J\\ 0 & -K_t/L & -R_a/L\end{pmatrix}\underline{x} + \begin{pmatrix}0\\ 0\\ \frac{1}{L}\end{pmatrix}v_a\\ \theta &= \begin{pmatrix}1 & 0 & 0\end{pmatrix}\underline{x}\end{aligned}$$

This is in the form

$$\begin{aligned}\underline{\dot{x}} =& A\underline{x} + Bu\\ y =& C\underline{x} + Du\end{aligned}$$

Note: FPE uses

$$\begin{aligned}\underline{\dot{x}} =& F\underline{x} + Gu\\ y =& H\underline{x}\end{aligned}$$